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Feedback control of uncertain systems: observers, dynamic compensators and adaptive stabilization

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FEEDBACK CONTROL OF UNCERTAIN SYSTEMS: OBSERVERS, DYNAMIC COMPENSATORS AND ADAPTIVE STABILIZATION

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ABSTRACT

This thesis considers aspects of deterministic control of uncertain dynamical systems, with particular reference to the design of observers, dynamic compensators and adaptive stabilization.

A major objective in deterministic theory is synthesis of feedbacks, based only on available knowledge of properties and bounds relating to the uncertainty, which guarantee that every member of the underlying class of uncertain systems exhibits some prescribed stability property. In achieving this objective, an assumption of full state measurement is frequently made; this is difficult to justify in practice where, generally, not all components of state can be measured. With the aim of relaxing this assumption, we consider two approaches to output-based design for classes of nominally linear uncertain systems.

In the first approach, we employ an observer to reconstruct the missing state components. The proposed control consists of a linear part to stabilize the nominal linear system and a nonlinear part to counteract uncertainties (nonlinear).

In the second approach, a dynamic output feedback control is proposed. Using a singular perturbation method, a threshold measure of "fastness" of the feedback dynamics, to ensure overall system stability, is derived. This threshold is calculable in terms of known bounds on the system uncertainties, but may be conservative in practice. To circumvent this drawback and to allow for bounded uncertainties with unknown bounds, an adaptive version of the proposed design is then developed.

The class of controls considered is extended to encompass discontinuous feedback which is modelled by an appropriately chosen set-valued map and the feedback controlled system is interpreted as generalized dynamical system. By using this formulation, we can enlarge the class of allowable uncertainties.

Finally, a class of "relative degree two" systems is considered as a special case of our general dynamic output feedback design. It is shown that this special class of systems can be stabilized by a static output feedback.

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CONTENTS

CHAPTER 1

DETERMINISTIC CONTROL OF UNCERTAIN DYNAMICAL SYSTEMS

1.1 Introduction

The process of mathematically modelling a physical dynamical system, in order to predict or to control its behaviour, generally induces some degree of imprecision or uncertainty. Typical uncertainties in the model are internal parameters, possibly time-varying, which are unknown or imperfectly known; uncertainties in the input (i.e. extraneous disturbances impinging on the system); and uncertainties in the state (i.e. measurement errors). These so-called *uncertain dynamical systems* have attracted much research recently, see for example, Gutman and Leitmann (1976a, b), Leitmann (1977, 1979b, 1980, 1981), Gutman (1979), Molander (1979), Corless and Leitmann (1981, 1983, 1984), Thorp and Barmish (1981), Gutman and Palmor (1982), Barmish and Leitmann (1982), Barmish, Corless and Leitmann (1983), Ryan (1983), Slotine and Sastry (1983), Balestrino *et al.* (1984), Ryan and Corless (1984), Ambrosino *et al.* (1985), Barmish (1985), Chen (1986a, 1988), Petersen and Hollot (1986), Chen and Leitmann (1987), Corless (1987), Goodall and Ryan (1988), and bibliographies therein.

With view to designing controllers for such uncertain systems, there are essentially two main approaches available to designer. The first approach is *stochastic control theory*, which is appropriate if *a priori* statistical characterization of the uncertainties in the system dynamics are available (e.g., see Åström

1970). The second approach is *deterministic control theory*, which is appropriate in cases for which the available information takes the form of known functional properties and bounds relating to the uncertain elements in the model.

Within the deterministic framework, one seeks feedback control which attempts to guarantee certain behaviour in the presence of uncertain information in the sense that *every* possible trajectory of the uncertain systems exhibits the desired behaviour. This desired behaviour is frequently asymptotic stability or ultimate boundedness.

Techniques of deterministic control in the presence of uncertainty separate into two categories. One category is *variable structure systems theory*, which developed initially in the USSR (see e.g. Itkis (1976), Utkin (1977, 1978)). This theory is based on the concept of an "attractive" design manifold, in the sense that neighbouring system trajectories are drawn onto the manifold and subsequently constrained to remain thereon. In addition, variable structure concepts are usefully employed in systems with uncertain and time-varying parameters in view of the invariance properties of "sliding modes" (Drazenović 1969). The second approach is *Lyapunov-based theory* developed by Leitmann and others, which originated in differential games analysis (see e.g., Leitmann 1976, Gutman and Leitmann 1976, Gutman 1979). In essence, this approach is based on the construction of a Lyapunov-type function V for the nominal system (i.e. the system in the absence of uncertainty). The controllers are synthesized such that they guarantee negativity of the time derivative of V along the solutions of the uncertain system under the "worst case" uncertainty. Once a controller has been generated, it guarantees the stability of the feedback system for all admissible uncertainty, since it is initially designed based on a "worst case" assumption. This design is sometimes called "the Lyapunov min-max" design (Gutman 1979). These two approaches, although historically distinct, are in fact, closely related. It has been shown (Ryan 1983, Ryan and Corless 1984 and Goodall and Ryan 1988) that the strengths of both theories could be exploited in a unified design which guarantees global uniform asymptotic stability or global uniform ultimate boundedness of a class of the feedback systems with bounded uncertainties.

It is often convenient when designing feedback control systems to assume initially that the full state of the system to be controlled is available through measurement. Thus, one might design a state feedback control law which can be implemented on the system. This is, for example, the control law that results from solution of a linear quadratic problem, from pole assignment problem, and from numerous other techniques that ensure stability and in some sense improve system performance. This state feedback approach has been successfully adopted by many researchers in the context of deterministic control of uncertain systems, see for example, Leitmann and others and their bibliographies, in the references cited above. Of particular interest are the approaches of Corless and Leitmann (1981) and Barmish, Corless and Leitmann (1983). In the former, it was shown that there exists a class of continuous state feedback controls which guarantee that every response of the system is uniformly ultimately bounded within an arbitrary small neighbourhood of the zero state. While in the latter, it was shown that the controller can be selected to be a linear time-invariant feedback of the state when the nominal system dynamics happen to be linear time-invariant. Moreover, it was illustrated by an example that a linear stabilizing controller can sometimes be constructed even when the system dynamics are nonlinear.

In general, however, not all states are available for measurement. This may be due to various technical reasons, for example, the measurement is too expensive, or it is strictly impossible to measure all the states. As a result, the feedback control law cannot be implemented. If that is the case, i.e. if only some states are measurable, an output-based controller is desirable. In the underlying principle of output feedback design, one has to use either "direct methods" or "indirect methods". A direct method is usually a "new" approach that directly accounts for inaccessibility of the entire state. Among papers written on stabilization of uncertain systems via static output feedback are Steinberg and Corless (1985) and Chen (1987c). Meanwhile, in the indirect method, one has to determine a suitable approximation to the state that can be incorporated in the feedback law. In essence, this approach results in a decomposition of the control design problem into two phases. The first phase is design of the control law assuming that the full state is available. This may be based on optimization or other design techniques and typically results in a control law without dynamics. The second phase is the design of a system that produces an approximation to the state. This system is called an *observer*, and was first developed by Luenberger (1964). Since then, observer theory has been extended by several researchers to include time-varying systems, discrete systems, and stochastic systems (see e.g. Luenberger 1971 and O'Reilly 1983). For feedback control of uncertain systems, observer-based design can be found in, for example, Leitmann (1981), Breinl and Leitmann (1983), Galimidi and Barmish (1986), Barmish and Galimidi (1986), Chen (1986b, 1987d) and Schmitendorf (1988c).

One of the fundamental issues in stabilization of uncertain systems is: what *a priori* assumptions must be imposed on the manner in which the uncertainties enter structurally into the state equations in order to guarantee stabilizability. In the cases of many previous references, these assumptions were known as *matching conditions*. These conditions have been exploited extensively in the literature dealing with stabilization using full state feedback, see e.g., Leitmann (1977, 1980), Gutman (1979), Corless and Leitmann (1981).

Many attempts have been made to relax these conditions to some extent. For example, in Leitmann and Barmish (1982), it is shown that ultimate boundedness is still possible as long as a measure of mismatch does not exceed a threshold limit; in Thorp and Barmish (1981), these matching conditions are somewhat generalized leading to a weaker requirement on the system structure; also in Molander (1979), the structure of the uncertainty was constrained by subspace relationships, in which its essentially play the role of matching conditions; and recently, Chen and Leitmann (1987) generalized the threshold mismatch by introducing the notion of "mismatch envelope".

A second fundamental issue is the question of robustness with respect to neglected dynamics. Suppose that a system consists of two subsystems, i.e. slow and fast dynamics. A desired property is derived for *reduced-order* system (i.e. a system in the absence of fast dynamics). The question then to be considered is essentially that of robustness with respect to neglected dynamics, viz. how does the presence of fast dynamics affect the performance of the feedback controlled uncertain system. It has been shown (Leitmann *et al.* 1986, Leitmann and Ryan 1987, Corless 1987 and Corless *et al.* 1989) that, under appropriate assumptions, the desired property of the reduced-order system is structurally stable in the sense that it is qualitatively retained by the full system provided that the neglected dynamics are sufficiently fast. Related questions of robustness are addressed in, for example, Khalil (1981, 1984), Young and Kokotović (1982), Kokotović (1985), Vidyasagar (1985), O'Reilly (1986), Garofalo (1988) and Linnemann (1988).

In the approach popularly known as *adaptive control*, controller parameters are adjusted continuously according to an adaptation law. A survey of the adaptive control theory and its applications through 1970s was given by Åström (1983). The research in the 1980s started by focusing on the robustness of adaptive schemes with respect to disturbances and unmodelled dynamics (see discussion by Kokotović 1985). One active area of research in adaptive control recently is called *universal adaptive stabilization.* These type of stabilizers are popularly known as "Nussbaum" controllers (Nussbaum 1983). Their application to minimum phase plants of relative degree one with unknown high-frequency gains was analyzed by Willems and Byrnes (1984), Mudgett and Morse (1985), Owens *et al.* (1987), Logemann and Owens (1988), and many others. The emphasis in this new work, essentially is the problem of reducing *a priori* information requirements. That is, the issue of concern is to determine the extent to which one can relax requirements such as that the plants degree and relative degree are known, the plant is minimum phase, and the sign of high-frequency gain is known. This research has culminated in necessary and sufficient conditions for universal adaptive stabilization (see, Byrnes *et al.* 1986, Mårtensson 1986).

This thesis is concerned with the problem of designing an output stabilizing controller for several classes of uncertain systems. Our study is restricted to linear time-invariant nominal systems. In the context of the above discussion, we will be looking at both methods (i.e. direct and indirect) and adaptive control. The precise formulation will be given in the next section.

1.2 Problem formulation

In this section, we formulate the general class of uncertain systems to be studied.

We consider uncertain nonlinearly perturbed linear systems of the general form

$$\dot{x}(t) = Ax(t) + Bu(t) + F(t, x(t), u(t)) \,, \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and F is an unknown function from the set $\mathcal{F}$ of all admissible perturbations to the system. We assume also that the only available state information is given by the output

$$y(t) = Cx(t) + \omega(t), \tag{1.2}$$

where $y(t) \in \mathbb{R}^p$ $(m \leq p \leq n)$, and $\omega(t) \in \mathbb{R}^p$ is bounded measurement noise. The triple (C, A, B) defines a nominal system (i.e. system in absence of uncertainty).

The problems studied (in general) may be stated as follows:

(i) Observer-based design (Indirect Method)

The objective is to design an observer-based feedback control law, i.e. to determine a Carathéodory function $\hat{u}$: $\mathbb{R}^n \to \mathbb{R}^m$ such that the control

$$u(t) = \hat{u}(\hat{x}(t)) \tag{1.3}$$

where $\hat{x}$ is an estimate of the state x, guarantees that, for each uncertainty realization $F \in \mathcal{F}$, the zero state of (1.1,1.2) with control (1.3) is ultimately bounded with respect to an "acceptably small" neighbourhood S of the zero state, in the sense that the state enters and remains within S after a finite interval of time.

(ii) Compensator-based design (Direct Method)

The objective is to design a dynamic compensator-based feedback control law, i.e. to determine Carathéodory functions f, φ: $\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^q \to \mathbb{R}^m$ such that the controller

$$\mu\dot{z}(t) = f(t, y(t), z(t)), \quad z(t) \in \mathbb{R}^q, \quad \mu > 0, \tag{1.4a}$$

$$u(t) = \varphi(t, y(t), z(t)) \tag{1.4b}$$

guarantees that, for each uncertainty realization $F \in \mathcal{F}$, the zero state of (1.1,1.2) with control (1.4) is globally uniformly asymptotically stable (in the sense of Lyapunov).

1.3 Design approaches, motivations and contributions

In order to achieve the objectives as given in § 1.2, we describe here the motivation of method of studies, the design approaches undertaken, and our main contributions to deterministic control of uncertain systems, particularly in design of observers, dynamic compensators and adaptive control. We present these under separate sub-titles, i.e. observer-based design, dynamic compensator-based design, adaptive-based design and static output-based design. We remark that each approach applies to a different class of systems.

1.3.1 Observer-based design

As we have mentioned in § 1.1, the observer-based design is based on an estimated state. The approach used is first to obtain a feedback control by assuming that the full state is available and then use an estimated state in the implementation of the controller. The estimated state is generated via a reduced-order observer which is based on the nominal system. This idea of using an observer based on the nominal system is due to Breinl and Leitmann (1983). The general feature of their approach is that the control consists of two parts, i.e. linear and nonlinear. The linear part is used to stabilize the nominal system, whereas the nonlinear part is designed to cope with uncertainties, i.e. it is designed to guarantee ultimate boundedness of the zero state in the presence of bounded uncertainties.

Our study is similar in principle to that of the above mentioned paper. we extend the approach to more general class of system uncertainties. Specifically, Breinl and Leitmann, consider only cone-bounded uncertainties whereas here we relax to non-cone-bounded, i.e. quadratically-bounded uncertainties.

Some previous works related to this observer-based design, can be found in, for example, Barmish and Galimidi (1986), Galimidi and Barmish (1986), Chen (1986b, 1987d) and Schmitendorf (1988c). However, except for Chen (1986b, 1987d), their designs are based on other approaches, e.g. based on "quadratic stabilizability" (see e.g. Barmish 1985) and a Riccati equation approach (see e.g. Petersen and Hollot 1986).

1.3.2 Dynamic compensator-based design

In this direct method, we propose a new dynamic output feedback control design for a class of uncertain systems. Our approach is similar in concept to that of Steinberg and Ryan (1986). The main feature of the approach is that the positive realness condition, required by the static output feedback design method of Steinberg and Corless (1985), is not imposed on the class of uncertain systems. Thus, our approach is applicable to a wider class of systems.

In essence, the approach is as follows. The control design is first carried out by considering a "hypothetical" output y_h for the system, to establish a stabilizing static output feedback control (which generally is unrealizable). This static control is then approximated by a realizable dynamic compensator (with parameter $\mu \geq 0$) which filters the actual system output y. Physically, the parameter μ is a measure of "fastness" for the filter dynamics; analytically, μ plays the role of a singular perturbation parameter. Using a singular perturbation analysis akin to that of Saberi and Khalil (1984) and Corless *et al.* (1989) (a detailed discussion of the use singular perturbation method as a tool to resolve many problems and its applications can be found in, e.g. Kokotović *et al.* 1986), a threshold measure μ^* of "fastness" of the compensator dynamics, to ensure overall system stability, is then derived. This threshold is calculable in terms of known bounds on the system uncertainties but corresponds to a "worst case" value it may be conservative in practice. To counteract this inherent conservatism and to allow for bounded uncertainties with unknown bounds, an adaptive version of the compensator is also developed (discussion in the next sub-section).

In this design, the main aims are threefold. First, to relax the minimum phase and relative degree 1 conditions of the nominal system. In Steinberg and Corless (1985), these conditions are imposed on the system, but here we only need that the "hypothetical" nominal system is minimum phase and relative degree 1. Thus, our system under consideration has relative degree ≥ 2; relative degree 1 turns out to be a special case. Secondly, to find a relationship (if any) between observer-based design and dynamic compensator-based designs. Thirdly, to generalize to more broader class of uncertain systems by admitting a discontinuous control. However, when a discontinuous control is coupled with system (1.1,1.2), the resulting system is governed by a differential equations with discontinuous right hand side. For such equations, the classical Carathéodory theory and concepts of solution are inappropriate. Consequently, the discontinuous feedback system is interpreted in the sense *generalized dynamical system* (see, e.g. Gutman 1979, Leitmann 1979), and defined via a *differential inclusion* (see, e.g. Aubin and Cellina 1984, Clarke 1983). This last aim (i.e. generalized feedback control) is achieved by adopting an approach that essentially of Ryan (1988). In order to include a more general class of system, i.e. to allow for unknown bounds with bounded uncertainties, the adaptive version to this design (i.e. generalized adaptive control), is also developed (discussion in the next sub-section).

1.3.3 Adaptive-based design

The design approach that has been described in § 1.3.2 will work well if we are given all information that fulfil the requirements of the design. We now consider the case for which bound on the uncertainties may be unknown. Recent developments in adaptive control of uncertain systems containing unknown functions with uncertain bounds has been made by Corless and Leitmann (1983, 1984).

Our design approach is also in similar spirit to that of Corless and Leitmann (1983, 1984), but it is developed by an approach which is essentially based on Mårtensson (1986). In that paper, he has used a rather weak assumption, viz. the order of any stabilizing regulator is sufficient *a priori* information for universal adaptive stabilization (see also, e.g. Byrnes *et al.* 1986). This adaptive version has a close relationship with compensator-based design that proposed in § 1.3.2, since it also has three aims. First, it is designed to counteract the inherent conservatism that results from crude estimates in "worst case" analysis. Secondly, to allow for bounded uncertainties with unknown bounds. Thus, this adaptive-based design may be regarded as an extension to the compensator-based design. Thirdly, to generalize to a more general class of uncertain systems, viz. by admitting a discontinuous control and analyzed in generalized sense of controlled differential inclusions (e.g. Aubin and Cellina 1984). We develop a generalized adaptive feedback control which follows that of Ryan (1988).

1.3.4 Static output-based design

In § 1.3.2, it was claimed that it is possible to design a stabilizing dynamic output feedback control for a class of uncertain systems, with "relative degree" ≥ 2. A natural question one might ask here is: is it possible to stabilize uncertain "relative degree 2" systems by using only static output feedback control?

We address in Chapter 6 the problem of designing static output feedback for a class of uncertain "relative degree 2" systems. This work has been motivated by our work developed in § 1.3.2 and the works of Steinberg and Ryan (1986) and Morse (1985). In Steinberg and Ryan (1986), as we have mentioned earlier, used a realizable dynamic compensator to stabilize a class of uncertain systems with relative degree 1 or 2. While, Morse (1985), has developed an universal controller which can adaptively stabilize any strictly proper, minimum phase system with relative degree not exceeding two.

However, in both above mentioned papers, they have only considered a class of single-input single-output systems. In Chapter 6, we extend it to multivariable case. It will be shown that we can design a static output feedback control for a class of uncertain systems, by imposing an extra or additional set of conditions on the system. Apart from the extra conditions, the procedure undertaken is similar to that used in § 1.3.2. Since it is designed on "worst case" analysis, the proposed feedback control is expected to be conservative. Thus, an adaptive version of this feedback control is conjectured; however stability of this remains an open question.

1.4 Organization of the thesis

The main results are contained in Chapters 3 to 6. Apart from this introductory chapter, the thesis is organized as follows.

Chapter 2 reviews the fundamental mathematical concepts that serve as foundations for our work. This includes the existence solutions of ordinary differential equations and differential inclusions, Lyapunov's stability theory, structural properties of linear systems (i.e. controllability and observability), feedback concepts including generalized feedback, observer theory, singular perturbation theory and universal adaptive stabilization.

We present our first results in Chapter 3. In that chapter, we incorporate an observer in an output feedback law in order to stabilize a class of uncertain systems. This observer-based design is preceded by establishing the existence of a full-state feedback stabilizing control.

In Chapter 4, we address the problem of design of dynamic output feedback controls for a class of uncertain systems. Here, we propose a new method to handle the problem by using singular perturbation theory. The second part of the chapter constitutes a generalization of the above proposed control design by admitting a discontinuous control component, modelled by an appropriately chosen set-valued map and interpreted in the generalized sense of a controlled differential inclusion.

Our proposed controller presented in Chapter 4 is designed by adopting a "worst case" analysis. Thus, the compensator is expected to be conservative in practice. To counteract this inherent conservatism and to allow for bounded uncertainties with unknown bounds, an adaptive version of the compensator is then developed in Chapter 5. Again, as in preceding chapter, the generalized adaptive control is developed by admitting a discontinuous control component modelled by a suitably chosen set-valued map.

Chapter 6 is devoted to a special class of uncertain systems known as "relative degree two" systems. We consider the possibility of stabilization of that special class by a static output feedback. A class of controllers indeed exists for this type of system by imposing an extra set of conditions on the nominal system. Since the "worst case" analysis is also adopted, the controller is expected to be conservative, and consequently an adaptive version is conjectured to allow for bounded uncertainties with unknown bounds and to circumvent the conservatism.

The thesis closes with Chapter 7, which gives summary and discussion of the results obtained, indicating some suggestions for future research and highlighting some possible extensions and applications.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

2.1 Introduction

The present chapter reviews the fundamental concepts that relate to our work. These ideas and concepts are presented to provide foundations and tools for our design and analysis. Since we are dealing with stabilization and compensation of a class of dynamical systems, the items of interest are: the existence of solutions of ordinary differential equations and differential inclusions, Lyapunov's stability theory, controllability and observability, feedback concepts, observer theory, singular perturbation theory and universal adaptive stabilization.

Since this material can be found in standard texts and research publications, we will not supply proofs for any of the results presented in this chapter.

2.2 Notation

In this section, we introduce notation which is used throughout the thesis.

Unless otherwise stated, small Roman or Latin letters will denote vectors, and capital Roman or Latin letters will denote matrices.

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^+ = [0, \infty)$. Let $\mathbb{R}^n$ be the set of ordered n-tuples of real numbers (Euclidean n-space). Let $x \in \mathbb{R}^n$, then $x = \text{col}\,(x_1, \cdots, x_n)$, i.e. x is presented as a column vector; and $x^T = (x_1, \cdots, x_n)$ denotes the transpose of x. Let $x, y \in \mathbb{R}^n$, then the function

$\langle\cdot,\cdot\rangle$: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an *inner product* and defined as follows:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \,.$$

Then we can define the function $\|\cdot\|$: $\mathbb{R}^n \to \mathbb{R}^+$, known as *Euclidean norm* induced by the inner product, which is given by

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}, \quad \text{for all } x \in \mathbb{R}^n \,.$$

Let $\mathbb{R}^{n\times m}$ be the space of all real $n\times m$ matrices. If $A = [a_{ij}] \in \mathbb{R}^{n\times m}$ is an arbitrary matrix, then A^T denotes the transpose of A. Now, let $A \in \mathbb{R}^{n\times n}$ be a square matrix. If A is non-singular, then A^{-1} denotes the inverse of A. The set of eigenvalues of A is denoted by $\sigma(A)$. If all its eigenvalues have negative real parts, we use $\sigma(A) \subset \mathbb{C}^-$, where $\mathbb{C}^-$ denotes the open left half the complex plane. If all eigenvalues of A happen to be real, we write $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ to denote the largest and smallest eigenvalues of A, respectively. The quadratic form associated with a square matrix A is denoted by $\langle x, Ax\rangle$.

If A is a diagonal matrix, we write $A = \text{diag}\,[a_1, \cdots, a_n]$. The identity matrix is denoted by I.

The norm of an arbitrary matrix A, induced by the Euclidean norm, is given by

$$\|A\| = [\sigma_{\max}(A^T A)]^{\frac{1}{2}} = [\max\,\{\lambda\colon \lambda \in \sigma(A^T A)\}]^{\frac{1}{2}} \,.$$

Let $\mathbb{B}_n(r)$ denotes the open ball of radius $r > 0$ centred at the origin in $\mathbb{R}^n$ (with closure $\overline{\mathbb{B}}_n(r)$), i.e.

$$\mathbb{B}_n(r) = \{x \in \mathbb{R}^n\colon \|x\| < r\} \,.$$

If $r = 1$, i.e. the open unit ball, we denotes it by $\mathbb{B}_n$.

For $S \subset \mathbb{R}^k$ and $z \in \mathbb{R}^k$, $z + S$ denotes the set $\{z + s\colon s \in S\} \subset \mathbb{R}^k$. For $S_1, S_2 \subset \mathbb{R}^k$, $S_1 + S_2$ denotes the set $\{s_1 + s_2\colon s_1 \in S_1;\ s_2 \in S_2\}$.

Finally, a remark about numbering of equations and theorems (including definitions, lemmas and corollaries): these are numbered in increasing order with the chapter indicated. For example, equation (3.2) means equation 2 of Chapter 3. Likewise, Theorem 5.4 means Theorem 4 of Chapter 5.

2.3 Solution concepts of ordinary differential equations and differential inclusions

The concept of solution for a given system is a fundamental issue to be addressed before proceeding to study the problem of stabilization or other problems. Of particular importance is the question of existence. Here, we summarize basic existence results for systems described by controlled ordinary differential equations and differential inclusions.

2.3.1 Ordinary differential equations

We consider a system governed by

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \tag{2.1a}$$

with initial value

$$x(t_0) = x_0, \tag{2.1b}$$

and bounded measurable input $u(\cdot)$.

A function x: $[t_0, \tau) \to \mathbb{R}^n$ will be said to be a *solution* of (2.1) if x is absolutely continuous and satisfies (2.1a) almost everywhere and (2.1b).

The following theorem provides conditions that suffice to guarantee the existence of solutions in respect with the requirements of our study. Before that we need the following definition.

Definition 2.1 *Carathéodory function*

A function f: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^p$ is Carathéodory iff:

(i) $f(\cdot, x, u)$ is Lebesgue-measurable for each fixed $(x, u) \in \mathbb{R}^n\times\mathbb{R}^m$;

(ii) $f(t, \cdot, \cdot)$ is continuous for each fixed $t \in \mathbb{R}$;

(iii) for each compact set $U \subset \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$, there exists a Lebesgue-integrable function $m_U(\cdot)$ such that

$$\|f(t,x,u)\| \leq m_U(t)\,, \quad \text{for all } (t,x,u) \in U\,.$$

Furthermore, if $m_U(\cdot) = m_U$, constant, then f is said to be *strongly* Carathéodory.

Now we state the existence theorem for ordinary differential equations (see Coddington and Levinson 1955).

Theorem 2.1 *The existence theorem of Carathéodory*

Let f: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ be Carathéodory. For each $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^n$ and bounded measurable $u(\cdot)$, the initial value problem (2.1) admits a solution.

Recall that the system (2.1a) is called *linear* if it is linear in x and u. Then it can be written as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2.2}$$

In most cases, system (2.2) arises from the "linearization" of system (2.1a). It is well known that, the general solution of (2.2) is given by the variation of parameters formula (see Coddington and Levinson 1955)

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)\, ds\,, \tag{2.3}$$

where $\Phi(t, t_0)$ is called the *transition matrix* of the system (2.2), with $\Phi(t_0, t_0) = I$. In case of (2.2) is linear time-invariant system, A and B are constants and equation (2.2) becomes

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.4}$$

and Φ is given by

$$\Phi(t, t_0) = \exp\,[A(t - t_0)]\,. \tag{2.5}$$

Almost in all parts of our study, we are dealing with this linear time-invariant system, since the design approach is based on this linear *nominal* system.

2.3.2 Differential inclusions

Before proceeding, we give the definition of a *set-valued map* or *multifunction.*

Definition 2.2

A *multifunction* Γ: $\mathbb{R}^m \to \mathbb{R}^n$ is a mapping from $\mathbb{R}^m$ to the subsets of $\mathbb{R}^n$. Thus, for each x in $\mathbb{R}^m$, $\Gamma(x)$ is a (possibly empty) set in $\mathbb{R}^n$.

The following definition is needed in connection with continuity of compact set-valued maps.

Definition 2.3

A compact-valued multifunction $\Gamma\colon \mathbb{R}^m \to \mathbb{R}^n$ is *upper semi-continuous* if it is upper semi-continuous at each $x \in \mathbb{R}^m$ in the following sense: given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\Gamma(x_1) \subset \Gamma(x) + \mathbb{B}_n(\varepsilon)$ for all $x_1 \in x + \mathbb{B}_m(\delta)$.

Consider again the system (2.1a). Suppose, for example, that the control takes the form of *discontinuous* state feedback. The resulting differential equation then has discontinuous right hand side, which renders the classical Carathéodory theory and concept of solution described in § 2.3.1 inappropriate. However, by embedding the feedback in a set-valued map $(t,x) \mapsto \mathcal{U}(t,x)$, the system may be interpreted in the sense of generalized dynamical systems (see, e.g., Gutman 1979, Leitmann 1979), and defined via a differential inclusion (see, e.g., Clarke 1983, Aubin and Cellina 1984). In fact, the theory of differential inclusions, extends many results from differential equations, such as those concerning the existence and nature of solutions, stability and invariance.

Thus, instead of considering system (2.1a), we now have to consider a differential inclusion

$$\dot{x}(t) \in \mathcal{G}(t, x(t)), \tag{2.6a}$$

$$x(t_0) = x_0 \tag{2.6b}$$

where $\mathcal{G}$ is a set-valued map defined as

$$\mathcal{G}(t,x) := \{f(t,x,u) : u \in \mathcal{U}(t,x)\}. \tag{2.7}$$

We will define precisely the set-valued map $\mathcal{G}$ in Chapters 4 and 5.

We now give a formal definition of solution of differential inclusion (2.6). A *solution* of (2.6) is defined to be an absolutely continuous function $x: [t_0, \tau) \rightarrow \mathbb{R}^n$ which satisfies (2.6a) almost everywhere and (2.6b).

The following theorem is sufficient for existence of a solution of a differential inclusion (Aubin and Cellina 1984, p. 98).

Theorem 2.2

Let $\Omega \subset \mathbb{R}\times\mathbb{R}^n$ be an open subset containing (t_0, x_0). Suppose that $\mathcal{G}: \Omega \rightarrow \mathbb{R}^n$ is a set-valued map with the properties:

(i) $\mathcal{G}$ is non-empty, compact and convex values;

(ii) $\mathcal{G}$ is upper semi-continuous.

Then there exists $\tau > 0$ and a solution $x(\cdot)$ of (2.6) defined on $[t_0, \tau)$.

2.4 Lyapunov's stability theory and related results

The present section is devoted in discussing concepts of stability according to Lyapunov. The direct or second method of Lyapunov is our essential tool in analysis of stability of given a system, and is frequently used in subsequent chapters.

2.4.1 The concepts of stability

A large variety of definitions of stability have been proposed; only those most suited to our need will be discussed in this section. To state these definitions, we return to the system (2.1a) again but now under feedback control.

Suppose we choose a continuous feedback control $u(t) = u(x(t))$. Then, with slight abuse of notation, system (2.1) has the form

$$\dot{x}(t) = f(t, x(t)), \tag{2.8a}$$

$$x(t_0) = x_0 \tag{2.8b}$$

Under Carathéodory assumption on f, then by Theorem 2.1, a local solution of (2.8) exists for each $(x_0, t_0) \in \mathbb{R}\times\mathbb{R}^n$.

Recall that a state x_e of the system (2.8) is said to be an *equilibrium state* if $f(t, x_e) = 0$, for all t. In other words, a motion passing through an equilibrium state at any time is actually at the same state at all future times. Any equilibrium state x_e can always be transferred to origin ($x \equiv 0$) by transformation $z = x - x_e$. Thus, without any loss of generality, we assume that the system (2.8) has $x_e = 0$ as an equilibrium state, with $f(t, 0) = 0$, for all t.

Assume further that the system (2.8) does not possess a finite escape times. Then, we state the following definitions of stability in the sense of Lyapunov.

Definition 2.4 *Stability*

The equilibrium state $x = 0$ of the system (2.8) is stable, if for any $\varepsilon > 0$ and t_0, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x_0\| \le \delta \Rightarrow \|x(t)\| \le \varepsilon, \quad \text{for all } t \ge t_0 .$$

Definition 2.5 *Attractivity*

The equilibrium state $x = 0$ of the system (2.8) is attractive, if there exists $\rho > 0$ and, to each $\eta > 0$ there corresponds a number $T_\rho(\eta, t_0)$ such that

$$\|x_0\| \le \rho \Rightarrow \|x(t)\| \le \eta, \quad \text{for all } t \ge t_0 + T_\rho(\eta, t_0) .$$

If ρ can be made arbitrarily large, then the equilibrium state $x = 0$ is said to be *globally attractive.*

In above definitions, if δ and T are independent of t_0, such stability is called *uniform.* Thus, we define the next important concept of stability.

Definition 2.6 *Global uniform asymptotic stability*

The equilibrium state $x = 0$ is called globally uniformly asymptotically stable if it is uniformly stable and globally uniformly attractive.

2.4.2 The direct method of Lyapunov

The direct method of Lyapunov attempts to deduce statements on the stability properties of equilibrium state of a system, without knowing its solution explicitly. This method actually has its origin from energy considerations. Lyapunov's idea was to generalize the energy arguments by introducing energy-like functions and evaluating their rate of change along the motion of the system under consideration. These functions are called *Lyapunov function candidates* for the system.

In short, the application of the direct method to stability problems consists of defining a Lyapunov function candidate with appropriate properties whose existence implies the desired type of stability. We state the global uniform asymptotic stability theorem for system (2.8) and define the class of Lyapunov functions for this case. By weakening various requirements on Lyapunov functions, we obtain other stability results as a by-product (see, Kalman and Bertram 1960).

Theorem 2.3 *(Lyapunov)*

Consider the system (2.8) with $f(t, 0) = 0$, for all t. Suppose there exists a function V: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ with continuous first partial derivatives with respect to t and x such that $V(t, 0) = 0$ and

(i) V is positive definite; i.e. there exists a continuous, monotonically increasing function α: $\mathbb{R}^+ \to \mathbb{R}^+$ such that $\alpha(0) = 0$, and for all t and $x \neq 0$

$$0 < \alpha(\|x\|) \leq V(t,x);$$

(ii) There exists a continuous function γ: $\mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma(0) = 0$ and for all t and all $x \neq 0$,

$$\mathcal{V}(t,x) := \frac{\partial}{\partial t} V(t,x) + \langle \nabla V(t,x), f(t,x) \rangle \leq -\gamma(\|x\|) < 0;$$

(iii) There exists a continuous, monotonically increasing function β: $\mathbb{R}^+ \to \mathbb{R}^+$ such that $\beta(0) = 0$, and for all t,

$$V(t,x) \leq \beta(\|x\|);$$

(iv) $\alpha(\|x\|) \to \infty$ with $\|x\| \to \infty$.

Then, the equilibrium state $x_e = 0$ is *globally uniformly asymptotically stable*. V is said to be a Lyapunov function for the system.

Corollary 2.1

The following conditions are sufficient for the various weaker types of stability:

(a) Uniform asymptotic stability: (i)-(iii).

(b) Uniform stability: (i), (iii) and (ii'): $\mathcal{V}(t,x) \leq 0$, for all t, x.

(c) Stability: (i)-(ii').

(d) No finite escape time: (i), (iv) and (ii''): $\mathcal{V}(t,x) \leq c_1 + c_2 V(t,x)$ for all t, x; c_1 and c_2 being positive constants.

In the case of linear time-invariant systems, we have the following result.

Corollary 2.2 *(Lyapunov)*

The equilibrium state $x_e = 0$ of the system

$$\dot{x}(t) = Ax(t) \tag{2.9}$$

is *asymptotically stable* if and only if, given any symmetric positive definite matrix Q there exists a symmetric positive definite matrix P which is the unique solution of the Lyapunov equation

$$PA + A^T P + Q = 0\,. \tag{2.10}$$

$V(x) = \langle x, Px \rangle$ is a Lyapunov function for the system (2.9).

2.4.3 Ultimate boundedness

In certain circumstances, the requirement of global uniform asymptotic stability (in the sense of Lyapunov) is too stringent. Hence, we relax it to *global uniform ultimate boundedness* with respect to some compact set S (which contains the zero state) in the sense that the state enters and remains thereafter within S after a finite interval of time. The following definition is due to Leitmann (1981) (see also, Corless and Leitmann 1981 and Barmish, Corless and Leitmann 1983).

Definition 2.7 *Global uniform ultimate boundedness with respect to $S \subset \mathbb{R}^n$*

The system (2.8) is said to be globally uniformly ultimately bounded with respect to the set $S \subset \mathbb{R}^n$ if:

(i) *existence of solutions*: for each $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^n$, there exists at least a solution x: $[t_0, t_1) \to \mathbb{R}^n$ of (2.8), with $x(t_0) = x_0$, $t_1 > t_0$;

(ii) *uniform boundedness*: given any $r > 0$, there exists $d(r) > 0$, such that for any solution x: $[t_0, t_1) \to \mathbb{R}^n$, $x(t_0) = x_0$ of (2.8),

$$\|x_0\| \le r \Rightarrow \|x(t)\| \le d(r)\,, \quad \text{for all } t \in [t_0, t_1);$$

hence, every such solution can be continued to a solution over $[t_0, \infty)$;

(iii) *uniform ultimate boundedness with respect to S*: given any $r > 0$, there exists $T(S, r) < \infty$, such that for any solution x: $[t_0, \infty) \to \mathbb{R}^n$, $x(t_0) = x_0$ of (2.8),

$$\|x_0\| \le r \Rightarrow x(t) \in S\,, \quad \text{for all } t \ge t_0 + T(S, r).$$

2.5 Feedback concepts

We discuss here the fundamental concepts of feedback design for linear (nominal) systems. We also summarized the state feedback control approaches for uncertain systems, on which our methods are based. First, we state the structural properties of feedback system, namely, the notions of controllability and observability.

2.5.1 Controllability and observability

(A) Controllability

Consider now a system given by (2.2). The main objective in feedback design is the regulation of the state $x(t)$ to some desired state, by chosen a suitable control input. The ability to exert the required control action is a structural characteristic of the system (2.2) known as *controllability.*

Recall that the system (2.2) is *completely controllable* if, for any t_0 and each $x_0 \in \mathbb{R}^n$, there exist $t_1 \geq t_0$ and control u: $[t_0, t_1] \to \mathbb{R}^m$ such that $x(t_1) = 0$.

For linear time-invariant system (2.4), we have a simple algebraic criterion for complete controllability.

The pair (A, B) is completely controllable if and only if rank $W_c = n$, where W_c is controllability matrix defined by

$$W_c := [B, AB, \cdots, A^{n-1}B] .$$

(B) Observability

Consider the system (2.2) again, but now with the output

$$y(t) = C(t)x(t) , \quad y(t) \in \mathbb{R}^p . \tag{2.11}$$

The concept of observability is concerned with the problem of determining the initial state, knowing only the output y for some interval of time. Formally, we may define this as follows.

The linear system (2.2) is said to be *completely observable* if, for any t_0, there exists $t_1 \geq t_0$ such that, each initial state $x(t_0) = x_0 \in \mathbb{R}^n$ can be uniquely determined from knowledge of the input u: $[t_0, t_1] \to \mathbb{R}^m$ and output y: $[t_0, t_1] \to \mathbb{R}^p$ functions.

Now let us define the matrix M (known as *observability Gramian*) given by

$$M(t_0, t) := \int_{t_0}^{t} \Phi^T(s, t_0) C^T(s) C(s) \Phi(s, t_0)\, ds \tag{2.12}$$

A stronger type of observability is obtained by imposing further conditions on the systems (see, e.g., Anderson 1977).

Definition 2.8 *Uniform complete observability*

The system (2.2) is uniformly completely observable if the following three conditions hold (any two implying the third): there exist $\tau > 0$ and positive constants $\alpha_i(\tau)$, $i = 1, \cdots, 4$, (which may depend on τ) such that for all s, t,

$$0 < \alpha_1(\tau) I \le M(t, t+\tau) \le \alpha_2(\tau) I \tag{2.13a}$$

$$0 < \alpha_3(\tau) I \le \Phi^T(t, t+\tau) M(t, t+\tau) \Phi(t, t+\tau) \le \alpha_4(\tau) I \tag{2.13b}$$

$$\|\Phi(t, s)\| \le \alpha_5(|t-s|) \tag{2.13c}$$

where function $\alpha_5 : \mathbb{R}^+ \to \mathbb{R}$ is bounded on bounded intervals and $\Phi(\cdot, \cdot)$ is the transition matrix generated by $A(\cdot)$.

Like controllability, for a linear time-invariant system, we have a simple algebraic criterion for complete observability.

The pair (C, A) is completely observable if and only if rank $W_o = n$, where W_o is observability matrix defined by

$$W_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

2.5.2 Continuous state feedback

In feedback design, the typical problem is the following: determine a function φ such that under the control $u(t) = \varphi(x(t))$ the system exhibits desired behaviour. First, we discuss the linear state feedback.

(A) Linear state feedback

Now consider linear time-invariant system (2.4). Suppose that its state $x(t)$ is completely accessible; then a linear feedback law of the form

$$u(t) = Kx(t) \tag{2.14}$$

can be applied to (2.4), results in the closed-loop system described by

$$\dot{x}(t) = (A + BK)x(t) \tag{2.15}$$

The state of (2.15) is asymptotically driven to the desired equilibrium state, if gain matrix K can be chosen such that the matrix $A + BK$ is stability matrix. The ability to do this is characterized by the following result (Wonham 1967).

Theorem 2.4

The pair (A, B) is controllable if and only if, for any symmetric set Λ of a complex numbers, there exists K such that $\sigma(A + BK) = \Lambda$.

The ability to assign any prescribed spectrum Λ is more than we require, since we seek only to determine K such that $\sigma(A + BK) \subset \mathbb{C}^-$.

Definition 2.9

The pair (A, B) is *stabilizable* if and only if, there exists K such that $\sigma(A+BK) \subset \mathbb{C}^-$.

(B) Continuous state feedback

Here, we give a summary of two control approaches to stabilize uncertain systems, which form the basis for constructing the output feedback control proposed in Chapters 3-6. We will discuss this in algorithmic form and in the context of our study, i.e. the design approach is based on a nominal linear system.

(i) Corless and Leitmann approach

This feedback design approach is proposed by Corless and Leitmann (1981):

- Choose K such that $\bar{A} := A+BK$ is a stability matrix, i.e. $\sigma(\bar{A}) \subset \mathbb{C}^-$.
- Solve Lyapunov equation

$$P\bar{A} + \bar{A}^T P + Q = 0 \tag{2.16}$$

for a given $Q > 0$. Then, $V(x) = \langle x, Px \rangle$ is a Lyapunov function.

- Form a continuous nonlinear control p: $\mathbb{R}^n \to \mathbb{R}^m$ as follows:

$$p(x) := \begin{cases} -\rho(x)\|B^T Px\|^{-1} B^T Px\,, & \text{if } \rho(x)\|B^T Px\| > \varepsilon \\ -\rho(x)\varepsilon^{-1} B^T Px\,, & \text{if } \rho(x)\|B^T Px\| \le \varepsilon \end{cases} \tag{2.17}$$

where $\varepsilon > 0$ is a prescribed constant (design parameter), and the function ρ: $\mathbb{R}^n \to \mathbb{R}^+$ is strongly Carathéodory, and determined via known bounds on the system uncertainties.

Then the control

$$u(t) = Kx(t) + p(x(t)) \tag{2.18}$$

stabilizes the uncertain system.

We will make use this approach in Chapter 3.

(ii) Barmish, Corless and Leitmann approach

This feedback design approach is made by Barmish, Corless and Leitmann (1983):

The first two steps are similar as in the above design approach. Then

- Form a control

$$u_\gamma(t) = -\gamma B^T P x(t)\,, \quad \gamma > 0\,, \tag{2.19}$$

and choose γ such that the corresponding Lyapunov derivative $\dot{\mathcal{V}}$ is negative.

Then for each fixed $\gamma > \gamma^*$, where γ^* is determined from known bounds on the system uncertainties, the control

$$u(t) = (K - \gamma B^T P)x(t) \tag{2.20}$$

stabilizes the uncertain system.

We will use the modification form of this type of control in Chapters 4-6.

2.5.3 Discontinuous state feedback

In section 2.3.2, we have discussed the concept of multifunction. Since we wish to admit a discontinuous control to stabilize the uncertain systems, a class of generalized feedbacks is defined.

Definition 2.10 *Generalized feedback*

A multifunction Ψ is a generalized feedback if:

(i) Ψ is upper semi-continuous with non-empty, convex and compact values;

(ii) Ψ is singleton-valued except on a set Σ_Ψ of (Lebesgue) measure zero.

For our purpose, we will employ a generalized output feedback control proposed by Ryan (1988). The control has a linear plus discontinuous output feedback structure of the form

$$u(t) \in -\hat{k}\left[(FCB)^{-1}Fy(t) + \mathcal{N}(y(t))\right] \tag{2.21a}$$

where $y \mapsto \mathcal{N}(y) \subset \mathrm{I\!R}^m$ is a set-valued map which, in essence models a discontinuous control component and is defined by

$$\mathcal{N}(y) := \begin{cases} \{\xi(y)\|(FCB)^{-1}Fy\|^{-1}(FCB)^{-1}Fy\}\,, & Fy \neq 0 \\ \bar{\mathrm{I\!B}}_m(\xi(y))\,, & Fy = 0 \end{cases} \tag{2.21b}$$

Then for each fixed $\hat{k} > \kappa^*$, where κ^* is determined by known bounds on the system uncertainties, the control (2.21) stabilizes the uncertain system, provided that $F \in \mathrm{I\!R}^{m\times p}$ exists such that FCB is known with $|FCB| \neq 0$, and ξ: $\mathrm{I\!R}^p \to \mathrm{I\!R}^+$ is a known continuous function. In this approach, the discontinuous control component is used to counteract an extra uncertainty component which is bounded by the function ξ of the system output y. Note further that, the nonlinear component of control is continuous everywhere except when $Fy = 0$ where it is discontinuous.

We will define this type of control precisely in Chapters 4-5.

2.6 Observer theory

In the previous section, we introduced state feedback under the assumption that the full state is available for measurement. This assumption often does not hold in practice, either because all state components are not accessible for direct measurement or because the number of measuring devices is limited. Thus, in order to apply state feedback to stabilize the system, we employ an observer that will estimate the missing state components, by utilizing the available inputs and outputs of the system.

2.6.1 Full-order observer

Consider linear system (2.4) with the output

$$y(t) = Cx(t)\,, \;\; y(t) \in \mathbb{R}^p\,. \tag{2.22}$$

Define an observer system given by

$$\dot{z}(t) = Dz(t) + Ey(t) + Hu(t)\,, \;\; z(t) \in \mathbb{R}^n\,, \tag{2.23}$$

where D, E and H are determined such that $z(t)$ is *asymptotic estimation* of a linear transformation $Tx(t)$, in the sense that if we define $e(t) = z(t) - Tx(t)$, then $e(t) \to 0$ as $t \to \infty$.

We first state the following general result.

Theorem 2.5

The state $z(t)$ in (2.23) is an asymptotic estimate of $Tx(t)$ for some constant $T \in \mathbb{R}^{n\times n}$ for any $(x_0, z_0) \in \mathbb{R}^n\times\mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ if and only if:

(i) $TA - DT = EC$;

(ii) $H = TB$;

(iii) $\sigma(D) \subset \mathbb{C}^-$.

As a special case, if $T = I$ in the above theorem, then the dynamic (2.23) is called a *full-order observer* or an *identity observer*. In this case, constraint (i) becomes $D = A - EC$. Thus, an identity observer is uniquely determined by selection of E. Relating to this issue, we have the following fundamental theorem.

Theorem 2.6

The pair (C, A) is observable if and only if, for any symmetric set Λ of n complex numbers, there exists E such that $\sigma(A - EC) = \Lambda$.

Definition 2.11

The pair (C, A) is *detectable* if and only if, there exists E such that $\sigma(A - EC) \subset \mathbb{C}^-$.

2.6.2 Reduced-order observer

The full-order observer we have just described above, although has simple structure, however possesses some redundancy. It stems from the fact that, while the observer constructs an estimate of the entire state, part of the state is already given by the available system outputs. This redundancy can be eliminated by building an observer of lower order but of arbitrary dynamics. This observer is called *reduced-order observer* or *minimal-order observer*.

The basic construction of a reduced-order observer is as follows. Since $y(t)$ has dimension p, an observer of order $(n-p)$ is constructed with state $z(t)$ that approximates $Tx(t)$ for some $p \times n$ matrix T, as in Theorem 2.5. Then an estimate $\hat{x}(t)$ of $x(t)$ can be determined through

$$\hat{x}(t) = \begin{bmatrix} T \\ C \end{bmatrix}^{-1} \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} \tag{2.24}$$

provided that the indicated inverse exists.

Suppose now the inverse of the matrix in (2.24) exists, then $\hat{x}(t)$ may be written as

$$\hat{x}(t) = S_1 z(t) + S_2 y(t) \tag{2.25}$$

Rewrite an observer (2.23) as

$$\dot{z}(t) = Dz(t) + Ey(t) + TBu(t)\,, \quad z(t) \in \mathbb{R}^q\,. \tag{2.26}$$

The following result is needed in Chapter 3, and is taken from Luenberger (1971) (see also Gopinath 1971).

Theorem 2.7

Define $\tilde{x}(t) := x(t) - \hat{x}(t)$. $\hat{x}(t)$ is an asymptotic estimation of state $x(t)$, i.e. $\tilde{x}(t) \to 0$ as $t \to \infty$ if and only if the following observer constraints are satisfied:

(i) $n-p \leq q \leq n$;

(ii) $TA - DT = EC$;

(iii) $S_1 T + S_2 C = I_n$;

(iv) $\sigma(D) \subset \mathbb{C}^-$.

2.6.3 State estimation and state feedback

Consider now the effect induced by using an estimated state (generated by an observer) in place of the actual value in the implementation of the control law. Of fundamental importance in this respect is the effect of introducing an observer on the closed-loop stability properties of the system. Fortunately, observers do not disturb stability properties when they are introduced.

It has been shown (Luenberger 1971) that, the eigenvalues of the composite system (i.e. feedback control and observer) are the union of those of state feedback (by assuming full state is available) and of observer. Thus, the *separation principle* is valid here. Consequently, the state feedback and observer can be designed independently. By combining the results of Theorems 2.4 and 2.6, and Definitions 2.9 and 2.11, we have Theorem 2.8 and Corollary 2.3 below.

Theorem 2.8

If the pair (A,B) is controllable and the pair (C,A) is observable with p linearly independent outputs, then for any symmetric set Λ of $(2n-p)$ complex numbers, there exists an observer of order $(n-p)$, such that the $(2n-p)$ eigenvalues of composite system can be set equal to Λ.

Corollary 2.3

If the pair (A,B) is stabilizable and the pair (C,A) is detectable with p linearly independent outputs, then there exists an observer of order $(n-p)$, such that $(2n-p)$ eigenvalues of composite system can be placed in open left half the complex plane.

2.7 Singular perturbation theory

In this section, we will briefly discuss what is known as the problem of singular perturbations and its relation to our study. The problem may be stated as follows.

Suppose we are given the system of nonlinear differential equations (known as a nonlinear singularly perturbed system)

$$\dot{x}(t) = f(t, x(t), z(t)), \quad x(t) \in \mathbb{R}^n, \quad z(t) \in \mathbb{R}^m, \tag{2.27a}$$

$$\varepsilon \dot{z}(t) = g(t, x(t), z(t)), \tag{2.27b}$$

where function f: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$, and function g: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^m$. Note that for any value of ε other than zero, the system (2.27) consists of $n+m$ differential equations. However, if $\varepsilon = 0$, then system (2.27) consists of n differential equations and m algebraic equations, because with $\varepsilon = 0$, (2.27b) reduces to

$$g(t, x(t), z(t)) = 0. \tag{2.28}$$

Now suppose it is possible to solve equation (2.28) to obtain an explicit expression for $z(t)$ in terms of $x(t)$, of the form

$$z(t) = h(t, x(t)), \tag{2.29}$$

where h: $\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$. Then (2.27a) with (2.29) reduce to

$$\dot{x}(t) = f(t, x(t), h(t,x(t))) \tag{2.30}$$

which is a system of n differential equations.

The parameter $\varepsilon = 0$ in (2.27b) is called a *singular perturbation parameter* because its value completely changes the nature of (2.27b), i.e. from a differential equation if $\varepsilon \neq 0$ to an algebraic equation if $\varepsilon = 0$. Briefly, the objective of singular perturbation theory is to examine the simplified system (2.30) and from this to draw conclusions about the original system (2.27) with $\varepsilon \neq 0$.

Related discussions of singular perturbation theory relevant to our work is given by Leitmann *et al.* (1986) and Leitmann and Ryan (1987) (see also

Kokotović *et al.* 1986 and O'Reilly 1986). Here, an equivalent theory is developed for the problem of robustness with respect to neglected dynamics. Thus, in context of this theory, $x(t)$ is the dominant or "slow" state, $z(t)$ is the state of the parasitic dynamics or "fast" state and $\varepsilon > 0$ is small scalar representing the parasitic elements (e.g., small inductances, capacitances, inertias, etc.). Neglecting the parasitic elements by setting $\varepsilon = 0$ in (2.27b), and substitution of $z(t)$ from (2.29) into (2.27a) yields the *reduced-order* system (2.30).

The robustness issue under discussion is whether a feedback control designed to stabilized the reduced-order (2.30), will in fact stabilize the actual system (2.27) for ε sufficiently small.

We will utilize this concept in Chapter 4.

2.8 Universal adaptive stabilization

In this final section, we will discuss briefly an approach of adaptive stabilization, popularly known as *universal adaptive stabilization.* Results to date show that there exist stabilizing adaptive control schemes of simple form, parameterized by a single gain parameter. Here, attention is restricted to the adaptive stabilization of first-order system by one-dimensional controllers and is taken from Byrnes *et al.* (1986).

Suppose Σ is a given class of linear systems (A, B, C) with (fixed) inputs and outputs, i.e.

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) \in \mathbb{R}^m, \tag{2.31}$$

$$y(t) = Cx(t), \quad y(t) \in \mathbb{R}^p. \tag{2.32}$$

By a *smooth controller* we mean a C^∞ system

$$\dot{z}(t) = f(z(t), y(t)), \quad z(t) \in \mathbb{R}^q, \tag{2.33}$$

$$u(t) = g(z(t), y(t)). \tag{2.34}$$

Definition 2.12 *Universal adaptive stabilizer*

A smooth controller is an universal adaptive stabilizer for Σ, provided that for each fixed system $(A, B, C) \in \Sigma$ and for all initial conditions $(x_0, z_0) \in \mathbb{R}^n \times \mathbb{R}$, the closed-loop system (2.31-2.34) satisfies:

(i) $\lim_{t \to \infty} x(t) = 0$;

(ii) $\lim_{t \to \infty} z(t) = z_\infty$.

Remark

Helmke and Prätzel-Wolters (1988) have considered a more general adaptive stabilizers. There, dynamic controllers may belong to some function space, i.e. analytic and piecewise continuous functions. Moreover, condition (ii) is relaxed to

(iia) there exists $M > 0$ such that $|z(t)| \le M$ for all $t \in [0, \infty)$.

In context of our study, we will use this approach in Chapters 5-6, and equations (2.34) and (2.33) is replaced respectively by (as it used in Byrnes *et al.* 1984 and Ilchmann *et al.* 1987)

$$u(t) = -k(t)y(t), \tag{2.35}$$

$$\dot{k}(t) = \|y(t)\|^2, \quad k(t) \in \mathbb{R}. \tag{2.36}$$

Hence, the condition (ii) is replaced by

$$\lim_{t \to \infty} k(t) = k_\infty < \infty .$$

CHAPTER 3

OBSERVERS FOR A CLASS OF UNCERTAIN SYSTEMS

3.1 Introduction

In this chapter, we present an observer-based design approach for stabilization of a class of uncertain systems. The aim of our study is the construction of an observer-based feedback control which guarantees that the response of the system enters and remains within a particular neighbourhood of the zero state after a finite interval of time.

The controller design adopted here is based on the approach of Breinl and Leitmann (1983). A salient feature of this approach is that the control consists of two parts, i.e. linear and nonlinear. The linear part is used to stabilize the nominal linear system, while the nonlinear part is designed to cope with uncertainties. We attempt to extend the approach to include a more general class of systems, by widening the class of allowable uncertainties; this will be precisely stated in the next section.

Although an observer-based controller design is our aim, we first establish the existence of a stabilizing state feedback control by assuming that the entire state is available for measurement. This is presented in § 3.3. Section 3.4 contains the second stage of the design procedure, wherein we employ a reduced-order observer for state estimation, and then implement the control by feeding back this estimated state. Under appropriate assumptions on the uncertainties, it will be shown that it is possible to design the feedback control and observer separately.

3.2 Problem statement and assumptions

The uncertain systems to be studied are governed by a differential equation of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + F(t, x(t), u(t)), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m \quad (3.1)$$

with an output equation is given by

$$y(t) = Cx(t) + \omega(t), \quad y(t) \in \mathbb{R}^p \quad (3.2)$$

where $m, p \leq n$, F is unknown function from the set $\mathcal{F}$ of all admissible uncertainty in the system and $\omega(t)$ is bounded measurement noise. The triple (C, A, B) which defines a nominal linear system is assumed to be known and satisfies the following assumptions:

A3.1: The pair (A, B) is stabilizable and B has full rank m.

A3.2: The pair (C, A) is observable and C has full rank p.

Next we impose some structural properties on uncertain function F, which implicitly define the set $\mathcal{F}$.

A3.3: F: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ is a Carathéodory function and satisfies the "matching conditions", i.e. there exists an unknown Carathéodory function g: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^m$ such that $F(\cdot) = Bg(\cdot)$ and g satisfies

$$\|g(t, x, u)\| \leq \gamma_0 + \gamma_1 \|x\| + \gamma_2 \|x\|^2 + \beta \|u\| . \quad (3.3)$$

β and γ_i $(i = 0, 1, 2)$ are known constants with

$$\beta < 1 \quad (3.4)$$

Remark

In Breinl and Leitmann (1983), the condition $\|g(t,x,u)\| \le \gamma\|x\| + \beta\|u\|$ was imposed on the uncertainty. Here, we relax it to (3.3), hence generalize their work.

Now, we state the problem to be studied which consists of two objectives. The first is that of designing a full state feedback control law, i.e. we would like to determine a Carathéodory function $u_0 \colon \mathbb{R}^n \to \mathbb{R}^m$ such that the control

$$u(t) = u_0(x(t)) \tag{3.5}$$

guarantees that, for each uncertainty realization $F \in \mathcal{F}$, the state of closed-loop system (3.1) and (3.5) is globally uniformly ultimately bounded with respect to a compact set S_0 containing the zero state (in the sense of Definition 2.7); this will be established in Theorem 3.1. Since (3.5) is unrealizable in general (in view of (3.2)), the second objective is that of designing an observer-based feedback control law, i.e. we would like to determine a Carathéodory function $u_1 \colon \mathbb{R}^n \to \mathbb{R}^m$ such that the control

$$u(t) = u_1(\hat{x}(t)) \tag{3.6}$$

where $\hat{x}(t)$ is an estimate of the state $x(t)$, guarantees that, for each uncertainty realization $F \in \mathcal{F}$, the state of closed-loop system (3.1-3.2) and (3.6) is ultimately bounded with respect to a compact set S_1 containing the zero state in the sense that the state enters and remains thereafter within set $\bar{S}_1$ after a finite interval of time; this will be established in Theorem 3.2.

3.3 Stabilization via full-state feedback

In this section, we present the first stage of our design. Assume now that the full state is accessible. Under the assumptions A3.1-A3.3, we will show that there exists a stabilizing state feedback control for this class of uncertain systems.

Following Breinl and Leitmann (1983), we split the control $u(t)$ into two parts, i.e.

$$u(t) = u_l(t) + u_n(t) \tag{3.7}$$

where $u_l(t)$ is the linear part and $u_n(t)$ is the nonlinear part. In what follows, we describe the control design procedure for both parts.

(i) Linear control part

This part is merely a linear control, i.e. it is of the form

$$u_l(t) = -Kx(t)\,. \tag{3.8}$$

We design this part to stabilize the nominal linear system, i.e. we want to choose gain matrix K such that $\sigma(A - BK) \subset \mathbb{C}^-$.

It is well known from the linear quadratic optimization problem (e.g., Kwakernaak and Sivan 1972) that, in view of A3.2, there exists a feedback control

$$u_l(t) = -B^T P x(t) = -Kx(t)\,, \tag{3.9}$$

where $P > 0$ is the unique symmetric positive definite solution of the Riccati equation

$$PA + A^T P - 2PBB^T P + Q = 0 \tag{3.10}$$

for a given $Q > 0$, which stabilizes the nominal system, i.e. $\sigma(A - BB^T P) \subset \mathbb{C}^-$.

(ii) Nonlinear control part

The nonlinear control part is designed to cope with the uncertainties and to guarantee stability of the closed-loop system in the presence of uncertainties. The construction of this control is based on Corless and Leitmann (1981), thus we use

$$u_n(t) = p(x(t)) \tag{3.11a}$$

where the function $p\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$p(x) := \begin{cases} -\rho(x)\|Kx\|^{-1}Kx, & \text{if } \rho(x)\|Kx\| > \varepsilon \\ -\rho^2(x)\varepsilon^{-1}Kx, & \text{if } \rho(x)\|Kx\| \leq \varepsilon \end{cases} \tag{3.11b}$$

where $P > 0$ is the solution of the Riccati equation (3.10) and function $\rho\colon \mathbb{R}^n \to \mathbb{R}^+$ is defined as

$$\rho(x) := (1-\beta)^{-1}\left[\gamma_0 + \gamma_1\|x\| + \gamma_2\|x\|^2 + \beta\|Kx\|\right] \tag{3.12}$$

Now we turn to the problem of constructing a full state feedback control which assures that, no matter what the uncertainties and initial conditions are, every solution of feedback controlled system is globally unifomly ultimately bounded with respect to a set S_0, to be specified in the sequel.

Suppose that the desired set S_0 of ultimate boundedness is specified as the closed ball of radius $\bar{d} > 0$ in $\mathbb{R}^n$, i.e.

$$S_0 = \bar{\mathbb{B}}_n(\bar{d}) \tag{3.13}$$

Define η_ε as

$$\eta_\varepsilon := \left[2\varepsilon\|Q^{-1}\|\right]^{\frac{1}{2}} \tag{3.14}$$

Our first task is to establish the following.

Theorem 3.1

Consider system (3.1), satisfying assumptions A3.1-A3.3 and under feedback control law (3.7), (3.9) and (3.11). For ε sufficiently small and for arbitrary uncertainty realization $F \in \mathcal{F}$, the feedback controlled system is globally uniformly ultimately bounded with respect to set S_0 (in the sense of Definition 2.7).

Proof

In view of A3.3 and control law (3.7), (3.9) and (3.11), the feedback controlled system can be written as

$$\dot{x}(t) = (A - BK)x(t) + Bp(x(t)) + Bg(t, x(t), -Kx(t) + p(x(t))) \quad (3.15)$$

Now we are going to prove the ultimate boundedness of (3.15) in several steps (in accordance with Definition 2.7).

(i) Existence of solutions:

The Carathéodory assumption (A3.3) on the function F ensures that, given any initial condition $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^n$, there exists a local solution $x\colon [t_0, t_1) \to \mathbb{R}^n$ of system (3.15), with $x(t_0) = x_0$, for some $t_1 > t_0$.

(ii) Uniform boundedness:

Consider a solution $x\colon [t_0, t_1) \to \mathbb{R}^n$, $x(t_0) = x_0$, of (3.15) with $\|x_0\| \le r$. We want to prove that this solution is bounded and so does not possess a finite escape time; hence, every such solution can be extended to a solution over $[t_0, \infty)$.

Since $P > 0$, define C^1 function (Lyapunov function candidate) $V\colon \mathbb{R}^n \to \mathbb{R}^+$ as

$$V(x) = \tfrac{1}{2}\langle x, Px \rangle\,, \quad \text{for all } x \in \mathbb{R}^n\,. \tag{3.16}$$

Now, consider the associated function $\mathcal{V}\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ given by

$$\mathcal{V}(t,x) := \langle \nabla V(x), (A-BK)x + Bp(x) + Bg(t,x,-Kx+p(x)) \rangle$$

$$= \langle Px, (A-BK)x + Bp(x) + Bg(t,x,-Kx+p(x)) \rangle \tag{3.17}$$

Then, in view of (3.3) and (3.10),

$$\mathcal{V}(t,x) \leq -\tfrac{1}{2}\langle x, Qx \rangle - \|Kx\|\|p(x)\| + \rho(x)\|Kx\|$$

Now, from Rayleigh's principle (Franklin 1968),

$$\sigma_{\min}(Q)\|x\|^2 \leq \langle x, Qx \rangle \leq \sigma_{\max}(Q)\|x\|^2$$

or, equivalently,

$$\|Q^{-1}\|^{-1}\|x\|^2 \leq \langle x, Qx \rangle \leq \|Q\|\|x\|^2 \tag{3.18}$$

Thus, in view of (3.12), (3.18): if $\rho(x)\|Kx\| > \varepsilon$,

$$\mathcal{V}(t,x) \leq -\tfrac{1}{2}\|Q^{-1}\|^{-1}\|x\|^2$$

and if $\rho(x)\|Kx\| \leq \varepsilon$,

$$\mathcal{V}(t,x) \leq -\tfrac{1}{2}\|Q^{-1}\|^{-1}\|x\|^2 + \varepsilon\,.$$

Consequently, for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$,

$$\mathcal{V}(t,x) \leq -\tfrac{1}{2}\|Q^{-1}\|^{-1}\|x\|^2 + \varepsilon\,. \tag{3.19}$$

Hence,

$$\mathcal{V}(t,x) < 0, \quad \text{for all } (t,x) \in \mathbb{R}\times(\mathbb{R}^n \backslash \overline{B}_n(\eta_\varepsilon)) \tag{3.20}$$

where η_ε is defined as in (3.14).

Now, along every solution $x\colon [t_0, t_1) \to \mathbb{R}^n$ of the feedback system,

$$\dot{V}(x(t)) = \mathcal{V}(t, x(t)) \quad \text{a.e.} \tag{3.21}$$

from which, together with (3.20), uniform boundedness is assured by selecting a function $d\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$d(r) := \begin{cases} [\|P\|\|P^{-1}\|]^{\frac{1}{2}}\eta_\varepsilon\,, & \text{if } r \leq \eta_\varepsilon \\ [\|P\|\|P^{-1}\|]^{\frac{1}{2}}r\,, & \text{if } r > \eta_\varepsilon \end{cases} \tag{3.22}$$

which yields

$$\|x(t)\| \leq d(r)\,, \quad \text{for all } t \geq t_0\,. \tag{3.23}$$

Therefore, every local solution $x(\cdot)$ is bounded and hence does not possess a finite escape times. Thus, every such solution can be extended into a solution over any compact interval, and hence, over $[t_0, \infty)$.

(iii) Uniform ultimate boundedness:

Let $x\colon [t_0, \infty) \to \mathbb{R}^n$, $x(t_0) = x_0$, be a solution of (3.15) with $\|x_0\| \leq r$. We want to show that there exists a finite $T(\bar{d}, r) > 0$ such that $\|x(t)\| \leq \bar{d}$, for all $t \geq t_0 + T(\bar{d}, r)$.

Now choose $\varepsilon > 0$ sufficiently small so that

$$\bar{d} > d(\eta_\varepsilon) = [\|P\|\|P^{-1}\|]^{\frac{1}{2}}\eta_\varepsilon\,,$$

where η_ε is defined by (3.14). Define $\bar{\eta}$ as

$$\bar{\eta} := [\|P\|\|P^{-1}\|]^{-\frac{1}{2}}\bar{d}\,. \tag{3.24}$$

Then, clearly $\bar{\eta} > \eta_\varepsilon$ and

$$d(\bar{\eta}) = \bar{d} \tag{3.25a}$$

or, equivalently,

$$\bar{\eta} = d^{-1}(\bar{d})\,. \tag{3.25b}$$

Now arguing as in Corless and Leitmann (1981), define $T(\bar{d}, r)$ as

$$T(\bar{d}, r) := \begin{cases} 0, & \text{if } r \le [\|P\|\|P^{-1}\|]^{-\frac{1}{2}}\bar{d} \\ c_0^{-1}[\|P\|r^2 - \|P\|^{-1}\|P^{-1}\|^{-2}\bar{d}^2], & \text{if } r > [\|P\|\|P^{-1}\|]^{-\frac{1}{2}}\bar{d} \end{cases} \tag{3.26}$$

with

$$c_0 := [\|Q^{-1}\|\|P\|\|P^{-1}\|]^{-1}\bar{d}^2 - 2\varepsilon \tag{3.27}$$

In view of (3.19) and uniform boundedness result (ii), global uniform ultimate boundedness property (iii) holds. Alternatively, it can be concluded that every solution x: $[t_0, \infty) \to \mathbb{R}^n$, with $x(t_0) = x_0$, of the feedback controlled system (3.15) must enters and thereafter remains within any closed ball containing a (Lyapunov) ellipsoid $\{x \in \mathbb{R}^n: V(x) \le \frac{1}{2}\|P\|\bar{\eta}^2\}$ which, in turn, contains the closed ball $\bar{\mathbb{B}}_n(\bar{\eta})$. One such candidate is the closed ball $\bar{\mathbb{B}}_n(\bar{d})$, with $\bar{d}$ given by (3.25a), since

$$\bar{\mathbb{B}}_n(\bar{d}) \supset \{x \in \mathbb{R}^n: V(x) \le \tfrac{1}{2}\|P\|\bar{\eta}^2\} \supset \bar{\mathbb{B}}_n(\bar{\eta}) .$$

Hence, the theorem has been established.

3.4 Observer-based controller

In the preceding section, we have established the existence of a stabilizing full state feedback control for the class of uncertain systems. To realize this, the full state must be available for feedback. However, in general situations, only some of the state components are available for measurement; the reason (as we have mentioned earlier) may be due to either that measuring devices are limited or that particular state components cannot be measured directly. Thus, we employ a reduced-order observer (Luenberger 1971) developed for a linear system as described in § 2.6.

Before proceeding, as a matter of convenience, we rewrite the observer equation (2.26) and state estimate (2.25) respectively as

$$\dot{z}(t) = Dz(t) + Ey(t) + TBu(t)\,, \quad z(t) \in \mathbb{R}^q\,, \tag{3.28}$$

and

$$\hat{x}(t) = S_1 z(t) + S_2 y(t) \tag{3.29}$$

where the observer (3.28) satisfies the "asymptotic estimation" constraints as given in Theorem 2.6, i.e.

(i) $n-p \le q \le n\,;$ (3.30a)

(ii) $TA - DT = EC\,;$ (3.30b)

(iii) $S_1 T + S_2 C = I_n\,;$ (3.30c)

(iv) $\sigma(D) \subset \mathbb{C}^-\,,$ (3.30d)

and so $\|\exp Dt\| \le Me^{-\delta t}$ for all $t \ge 0$ and for some known constants $M, \delta > 0$. Recall that (in absence of uncertainty), the matrices D, E, T, S_1 and S_2 are determined such that $z(t)$ is an asymptotic estimation of the linear transformation $Tx(t)$, i.e. if we define the estimation error $e(t)$ as

$$e(t) = z(t) - Tx(t) \tag{3.31}$$

then,

$$\lim_{t \to \infty} e(t) = 0\,.$$

Moreover, if and only if the constraints (3.30) are satisfied, then

$$\lim_{t \to \infty} (x(t) - \hat{x}(t)) = 0\,,$$

i.e. $\hat{x}(t)$ is an asymptotic estimation of state $x(t)$ in absence of uncertainty.

In our case, that is for the uncertain system (3.1), we impose additional structure on the uncertainty g.

A3.4: For all $(t, x, u) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$,

$$\|TBg(t,x,u)\| \leq \kappa_T\,, \tag{3.32}$$

where κ_T is a known constant.

Remark

Breinl and Leitmann (1983) imposed the stronger condition $TB = 0$. Here it is relaxed to (3.32).

To employ this reduced-order observer for the uncertain system (3.1,3.2), again we adopt an approach of Breinl and Leitmann (1983) where we use $u(t) = \hat{u}_l(t) + \hat{u}_n(t)$, i.e. we replace the state $x(t)$ by the estimate $\hat{x}(t)$, which results in control laws (3.9) and (3.11) respectively replaced by

$$\hat{u}_l(t) = -K\hat{x}(t) \tag{3.33}$$

and

$$\hat{u}_n(t) = \hat{p}(\hat{x}(t)) \tag{3.34a}$$

where the function $\hat{p}$: $\mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\hat{p}(\hat{x}) := \begin{cases} -\hat{\rho}(\hat{x})\|K\hat{x}\|^{-1}K\hat{x}\,, & \text{if } \hat{\rho}(\hat{x})\|K\hat{x}\| > \varepsilon \\ -\hat{\rho}^2(\hat{x})\varepsilon^{-1}K\hat{x}\,, & \text{if } \hat{\rho}(\hat{x})\|K\hat{x}\| \leq \varepsilon \end{cases} \tag{3.34b}$$

$$\hat{\rho}(\hat{x}) := (1-\beta)^{-1}\big[\gamma_0 + \gamma_1(\|\hat{x}\| + \tilde{\rho}_\varepsilon) + \gamma_2(\|\hat{x}\| + \tilde{\rho}_\varepsilon)^2$$

$$+ \beta\|K\hat{x}\| + \|K\|\tilde{\rho}_\varepsilon\big] \tag{3.34c}$$

where $\varepsilon > 0$ is a design parameter, $P > 0$ is the solution of the Riccati equation (3.10) for a given $Q > 0$, and the parameter $\tilde{\rho}_\varepsilon > 0$ will be defined later (in (3.43b)).

In order to proceed, we define state estimation error $\tilde{x}(t)$ as

$$\tilde{x}(t) := \hat{x}(t) - x(t) \tag{3.35}$$

and, in view of the state estimate (3.29) and observer constraint (3.30c), we have

$$\tilde{x}(t) = S_1 e(t) + S_2 \omega(t) \tag{3.36}$$

Since we are dealing with an asymptotic estimation, it is more convenient to consider the estimation error $e(t)$ rather than observer state $z(t)$. Thus, the overall observer-feedback controlled system, i.e. system (3.1) under control $\hat{u}(t)$ given by (3.33) and (3.34), which, in view of (3.36), can be expressed in the form

$$\begin{aligned} \dot{x}(t) &= (A-BK)x(t) + B\hat{p}(\hat{x}(t)) \\ &\quad + B[g(t,x(t),-K\hat{x}(t)+\hat{p}(\hat{x}(t))) - K\tilde{x}(t)] \end{aligned} \tag{3.37}$$

and, in view of (3.30b) and (3.31), we may write the error dynamic equation as

$$\dot{e}(t) = De(t) + E\omega(t) - TBg(t,x(t),-K\hat{x}(t)+\hat{p}(\hat{x}(t))) \tag{3.38}$$

Now, we impose additional assumptions on ω.

A3.5: The function ω: $\mathbb{R} \to \mathbb{R}^p$ is measurable and bounded, i.e.

$$\|\omega(t)\| \le \kappa_\omega\ , \quad \text{for all } t \in \mathbb{R}\,,$$

where κ_ω is a known constant.

We are going to investigate the ultimate boundedness property of (3.37,3.38). We will do this by initially proving existence and continuation of solutions of (3.37,3.38); this is proved in Lemma 3.1. Then, under the standing assumptions and two additional assumptions (one will be specified in A3.6 below and the other in due course), it is shown that ultimate boundedness of (3.37,3.38) is assured in a particular neighbourhood of the zero state.

Suppose $(x(\cdot), e(\cdot))$ is a solution of (3.37,3.38) (this is a valid assumption, since F is a Carathéodory and ω is measurable and bounded, and will be phrased precisely in Lemma 3.1). Now recall that since $\sigma(D) \subset \mathbb{C}^-$,

$$\|\exp D(t-t_0)\| \leq M e^{-\delta(t-t_0)}, \tag{3.39}$$

for all $t \geq t_0$ and for some M, $\delta > 0$. Define

$$\tilde{\rho}_e := \delta^{-1} M(\|E\| \kappa_\omega + \kappa_T) \tag{3.40}$$

then, in view of A3.4 and A3.5, along every solution $(x(\cdot), e(\cdot))$ of (3.37,3.38) we have

$$\|e(t)\| \leq \tilde{\rho}_e + e^{-\delta(t-t_0)}[M\|e(t_0)\| - \tilde{\rho}_e], \text{ for all } t \geq t_0. \tag{3.41}$$

Now define

$$\tilde{\rho} := \|S_1\| \tilde{\rho}_e + \|S_2\| \kappa_\omega \tag{3.42}$$

then, in view of (3.36), (3.41) and (3.42),

$$\|\tilde{x}(t)\| \leq \tilde{\rho} + c e^{-\delta(t-t_0)}, \text{ for all } t \geq t_0, \text{ and } c \text{ is a constant}, \tag{3.43a}$$

$$\leq \tilde{\rho} + \varepsilon =: \tilde{\rho}_\varepsilon \text{ for sufficiently large } t. \tag{3.43b}$$

Note that (3.42a) will be used in establishing of existence and continuation of solutions (Lemma 3.1), while (3.42b) will be used for ultimate boundedness (Theorem 3.2).

We now impose our final assumption.

$$\text{A3.6:} \quad \gamma_2 < \frac{(1-\beta)}{4\|Q^{-1}\|\|K\|\tilde{\rho}_\varepsilon} .$$

Before proceeding, we observe that for all $t \geq t_0$ the following holds

$$\begin{aligned}\|g(t, x(t), -K\hat{x}(t) + \hat{p}(\hat{x}(t))) - K\tilde{x}(t)\| \leq \gamma_0 &+ \gamma_1 \|x(t)\| + \gamma_2 \|x(t)\|^2 \\ &+ \beta\|K\hat{x}(t)\| + \beta\|\hat{p}(\hat{x}(t))\| \\ &+ \|K\|[\tilde{\rho} + ce^{-\delta(t-t_0)}] .\end{aligned}$$

From (3.35) and (3.43a),

$$\begin{aligned}\|x(t)\| &\leq \|\hat{x}(t)\| + \|\tilde{x}(t)\| \\ &\leq \|\hat{x}(t)\| + \tilde{\rho} + ce^{-\delta(t-t_0)} , \quad \text{for all } t \geq t_0 .\end{aligned}$$

Therefore

$$\begin{aligned}\|g(t, x(t), -K\hat{x}(t) + \hat{p}(\hat{x}(t))) - K\tilde{x}(t)\| \leq \gamma_0 &+ \gamma_1(\|\hat{x}(t)\| + \tilde{\rho}) + \gamma_2(\|\hat{x}(t)\| + \tilde{\rho})^2 \\ &+ \beta\|K\hat{x}(t)\| + \beta\|\hat{p}(\hat{x}(t))\| + \|K\|\tilde{\rho} \\ &+ \gamma_1 ce^{-\delta(t-t_0)} + \gamma_2 c^2 e^{-2\delta(t-t_0)} \\ &+ 2\gamma_2 ce^{-\delta(t-t_0)}(\|\hat{x}(t)\| + \tilde{\rho}) \\ &+ \|K\| ce^{-\delta(t-t_0)}\end{aligned}$$

Then, using (3.34c), we have

$$\begin{aligned}\|g(t, x(t), -K\hat{x}(t) + \hat{p}(\hat{x}(t))) - K\tilde{x}(t)\| \leq \hat{p}(\hat{x}(t)) \\ + ce^{-\delta(t-t_0)}[c_1 + c_2 e^{-\delta(t-t_0)} + c_3 \|\hat{x}(t)\|]\end{aligned} \tag{3.44}$$

where

$$c_1 := \gamma_1 + 2\gamma_2 \tilde{\rho},$$

$$c_2 := c\gamma_2,$$

$$c_3 := 2\gamma_2.$$

Now we establish the existence and continuation of solutions of system (3.37,3.38).

Lemma 3.1

Consider the composite feedback controlled system (3.37,3.38), satisfying A3.1-A3.5. For arbitrary uncertainty realization $F \in \mathcal{F}$ and for each $(t_0, x_0, e_0) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^q$, there exists a local solution $(x, e)\colon [t_0, t_1) \to \mathbb{R}^n\times\mathbb{R}^q$ of the feedback controlled system (3.37,3.38), with $(x(t_0), e(t_0)) = (x_0, e_0)$, for some $t_1 > t_0$. Moreover, every such solution can be continued into a solution over $[t_0, \infty)$.

Proof

In view of the Carathéodory assumption (A3.3) on F ensures that, for each $(t_0, x_0, e_0) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^q$, there exists a local solution $(x, e)\colon [t_0, t_1) \to \mathbb{R}^n\times\mathbb{R}^q$ of the feedback controlled system (3.37,3.38) with $(x(t_0), e(t_0)) = (x_0, e_0)$, for some $t_1 > t_0$.

To establish that every such solution can be extended into a solution over $[t_0, \infty)$, the behaviour (along local solutions of (3.37,3.38)) of the function $V(\cdot)$ (defined by (3.16)) is examined.

Consider now the associated function $\mathcal{W}$: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\mathcal{W}(t,x,\hat{x}) := \langle \nabla V(x), (A-BK)x + B\hat{p}(\hat{x})$$

$$+ B[g(t,x,-K\hat{x}+\hat{p}(\hat{x})) - K\tilde{x}]\rangle$$

$$= \langle Px, (A-BK)x + B\hat{p}(\hat{x})$$

$$+ B[g(t,x,-K\hat{x}+\hat{p}(\hat{x})) - K\tilde{x}]\rangle$$

where $\tilde{x} = x - \hat{x}$. In view of (3.10),

$$\mathcal{W}(t,x,\hat{x}) = -\tfrac{1}{2}\langle x, Qx\rangle + \langle Kx, \hat{p}(\hat{x})\rangle$$

$$+ \langle Kx, [g(t,x,-K\hat{x}+\hat{p}(\hat{x})) - K\tilde{x}]\rangle \tag{3.45}$$

Now in view of (3.35) and (3.44), along every local solution $(x(\cdot), e(\cdot))$ of (3.37,3.38),

$$\mathcal{W}(t,x(t),\hat{x}(t)) \le -\tfrac{1}{2}\langle x(t), Qx(t)\rangle - \|K\hat{x}(t)\|[\|\hat{p}(\hat{x}(t))\| - \hat{\rho}(\hat{x}(t))]$$

$$+ \|K\tilde{x}(t)\|[\|\hat{p}(\hat{x}(t))\| + \hat{\rho}(\hat{x}(t))]$$

$$+ \|K\hat{x}(t)\| c e^{-\delta(t-t_0)}[c_1 + c_2 e^{-\delta(t-t_0)} + c_3\|\hat{x}(t)\|]$$

$$+ \|K\tilde{x}(t)\| c e^{-\delta(t-t_0)}[c_1 + c_2 e^{-\delta(t-t_0)} + c_3\|\hat{x}(t)\|] \tag{3.46}$$

Since $\|\tilde{x}(t)\| \le \tilde{\rho} + c$ for all $t \ge t_0$, we have

$$\mathcal{W}(t,x(t),\hat{x}(t)) \le -\tfrac{1}{2}\langle x(t), Qx(t)\rangle + \varepsilon + 2\|K\|(\tilde{\rho}+c)\hat{\rho}(\hat{x}(t))$$

$$+ c\|K\hat{x}(t)\|[c_1 + c_2 + c_3\|\hat{x}(t)\|]$$

$$+ c\|K\|(\tilde{\rho}+c)[c_1 + c_2 + c_3\|\hat{x}(t)\|] \tag{3.47}$$

But from (3.35),

$$\|\hat{x}(t)\| \le \|x(t)\| + \|\tilde{x}(t)\|$$

$$\le \|x(t)\| + \tilde{\rho} + c$$

and so we can do the following estimation:

$$\hat{\rho}(\hat{x}(t)) \le a_1 + a_2\|x(t)\| + a_3\|x(t)\|^2 ,$$

$$\|K\hat{x}(t)\| \le a_4 + a_5\|x(t)\| ,$$

$$\|K\hat{x}(t)\|\|\hat{x}(t)\| \le a_6 + a_7\|x(t)\| + a_8\|x(t)\|^2 .$$

Thus, using these in (3.47) yields

$$\mathcal{W}(t, x(t), \hat{x}(t)) \le k_0 + k_1\|x(t)\| + k_2\|x(t)\|^2 , \tag{3.48}$$

for all $t \ge t_0$, where k_i, $i = 0, 1, 2$ are positive constants.

Now, along every solution $(x(\cdot), e(\cdot))$ of the feedback controlled system (3.37,3.38),

$$\dot{V}(x(t)) = \mathcal{W}(t, x(t), \hat{x}(t)) \quad \text{a.e.} \tag{3.49}$$

Thus, from the inequality

$$\tfrac{1}{2}\|P^{-1}\|^{-1}\|x\|^2 \le V(x) \le \tfrac{1}{2}\|P\|\|x\|^2 ,$$

(3.48) can be written as

$$\dot{V}(x(t)) \le k_0 + k_3V^{\frac{1}{2}}(x(t)) + k_4V(x(t)) \tag{3.50}$$

where $k_3 = k_1(2\|P^{-1}\|)$ and $k_4 = k_1(2\|P^{-1}\|)^{\frac{1}{2}}$. Using approximation $V^{\frac{1}{2}}(x) \le (1 + V(x))$, we have

$$\dot{V}(x(t)) \le \kappa_0 + \kappa_1V(x(t)) . \tag{3.51}$$

where $\kappa_0 = k_0 + k_3$ and $\kappa_1 = k_3 + k_4$.

Now, by invoking Corollary 2.1(d), we may conclude that every local solution $(x(\cdot), e(\cdot))$ of the feedback controlled system (3.37,3.38) does not possess a finite escape times. Thus, every such solution can be extended into a solution over any compact interval, and hence can be extended indefinitely. This completes the proof of lemma.

Let T be sufficiently large so that

$$\|\tilde{x}(t)\| \le \tilde{\rho}_\varepsilon \ , \quad \text{for all } t \ge T. \tag{3.52}$$

Then the following holds for all $t \ge T$,

$$\begin{aligned}\|g(t, x(t), -K\hat{x}(t) + \hat{p}(\hat{x}(t))) - K\tilde{x}(t)\| &\le \gamma_0 + \gamma_1(\|\hat{x}(t)\| + \tilde{\rho}_\varepsilon) + \gamma_2(\|\hat{x}(t)\| + \tilde{\rho}_\varepsilon)^2 \\ &\quad + \beta\|K\hat{x}(t)\| + \beta\|\hat{p}(\hat{x}(t))\| + \|K\|\tilde{\rho}_\varepsilon \\ &\le \hat{\rho}(\hat{x}(t)) \end{aligned} \tag{3.53}$$

Using

$$\begin{aligned}\|\hat{x}(t)\| &\le \|x(t)\| + \|\tilde{x}(t)\| \\ &\le \|x(t)\| + \tilde{\rho}_\varepsilon\end{aligned}$$

then

$$\begin{aligned}\hat{\rho}(\hat{x}(t)) &\le (1-\beta)^{-1}\big[\gamma_0 + \gamma_1(\|x(t)\| + 2\tilde{\rho}_\varepsilon) + \gamma_2(\|x(t)\| + 2\tilde{\rho}_\varepsilon)^2 \\ &\quad + \beta\|K\|(\|x(t)\| + \tilde{\rho}_\varepsilon) + \|K\|\tilde{\rho}_\varepsilon\big] \\ &= a + b\|x(t)\| + c\|x(t)\|^2 \ , \end{aligned} \tag{3.54}$$

where

$$a := (1-\beta)^{-1}[\gamma_0 + (2\gamma_1 + 4\gamma_2\tilde{\rho}_\varepsilon + \beta\|K\| + \|K\|)\tilde{\rho}_\varepsilon] \,,$$

$$b := (1-\beta)^{-1}[\gamma_1 + 4\gamma_2\tilde{\rho}_\varepsilon + \beta\|K\|] \,,$$

$$c := (1-\beta)^{-1}\gamma_2 \,.$$

Consider now the closed ball $\bar{\mathbb{B}}_n(\eta)$ of radius

$$\eta := \frac{2\|K\|\tilde{\rho}_\varepsilon}{\theta} b + 2\left[\left[\frac{\|K\|\tilde{\rho}_\varepsilon}{\theta}\right]^2 b^2 + \frac{\frac{\varepsilon}{2} + \|K\|\tilde{\rho}_\varepsilon a}{\theta}\right]^{\frac{1}{2}} \tag{3.55}$$

where

$$\theta := [\|Q^{-1}\|^{-1} - 4\|K\|\tilde{\rho}_\varepsilon c] \,. \tag{3.56}$$

Note that θ defined above is positive by virtue of A3.6 and definition of c in (3.54).

We now ready to state the main theorem of this chapter.

Theorem 3.2

For arbitrary uncertainty realization $F \in \mathcal{F}$, the feedback controlled system (3.37,3.38) which satisfies A3.1-A3.6 is ultimately bounded with respect to every Lyapunov ellipsoid which contains the closed ball $\bar{\mathbb{B}}_n(\eta)$ in its interior.

Proof

We consider again now the Lyapunov function $V(\cdot)$ defined by (3.16) and its associated function $\mathcal{W}(\cdot)$ introduce in Lemma 3.1. Thus, from (3.45) and in view of (3.53), along solutions $(x(\cdot), e(\cdot))$ of (3.37,3.38) the following holds for sufficiently large t,

$$\mathcal{W}(t, x(t), \hat{x}(t)) \le -\tfrac{1}{2}\langle x(t), Qx(t)\rangle + \varepsilon + 2\|K\|\tilde{\rho}_\varepsilon \hat{\rho}(\hat{x}(t)) \tag{3.57}$$

Using (3.18) and (3.54), we have

$$\mathcal{W}(t, x(t), \hat{x}(t)) \le -\tfrac{1}{2}\left[\theta\|x(t)\|^2 - 4\|K\|\tilde{\rho}_\varepsilon b\|x(t)\| - 2\varepsilon - 4\|K\|\tilde{\rho}_\varepsilon a\right] \tag{3.58}$$

where θ is defined by (3.56). Hence,

$$\mathcal{W}(t, x(t), \hat{x}(t)) < 0 \ , \quad \text{for all } (t, x) \in \mathbb{R}\times(\mathbb{R}^n \backslash \bar{\mathbb{B}}_n(\eta)) \tag{3.59}$$

where η is defined as in (3.55).

Now, along every solution $(x(\cdot), e(\cdot))$ of (3.37,3.38), (3.49) holds for sufficiently large t, from which, together with (3.59), we may conclude that every solution $(x(\cdot), e(\cdot))$ of the feedback controlled system (3.37,3.38) must ultimately enters and thereafter remains within any Lyapunov ellipsoid which contains the closed ball $\bar{\mathbb{B}}_n(\eta)$ in its interior, i.e. $S_1 = \{x \in \mathbb{R}^n : \tfrac{1}{2}\langle x, Px\rangle \supset \bar{\mathbb{B}}_n(\eta)\}$. This completes the proof of the theorem.

CHAPTER 4

DYNAMIC OUTPUT FEEDBACK STABILIZATION OF A CLASS OF UNCERTAIN SYSTEMS

4.1 Introduction

In the preceding chapter, we considered a problem of designing a dynamic output feedback control for a class of uncertain systems, which is based on the construction of an asymptotic Luenberger state observer. Here, we will consider another approach to dynamic output feedback control of uncertain systems, i.e. a direct method, which we called "dynamic compensator-based design". In this approach, we propose a new dynamic output feedback control design for a class of uncertain systems. Our approach is similar in concept to that of Steinberg and Ryan (1986), and fundamentally based on that of Barmish, Corless and Leitmann (1983) and Steinberg and Corless (1985).

The main feature of the approach is that the positive realness condition, required by the static output feedback design method of Steinberg and Corless (1985), is not imposed on the class of uncertain system. To be precise, Steinberg and Ryan (1986) have considered a stabilizing dynamic output feedback control for a class of single-input single-output uncertain systems whose nominal transfer functions have relative degree 2. It is our goal of this chapter to extend their approach to a class of multi-input multi-output uncertain systems.

In essence, the approach is as follows. Initially considering a hypothetical output y_h for the system, a (generally unrealizable) stabilizing static output feedback control is established. This static control is then approximated by a realizable dynamic compensator (with parameter $\mu > 0$) which filters the actual output y. Physically, the parameter μ is a measure of "fastness" for the filter dynamics; analytically, μ plays the role of a singular perturbation parameter. Using a singular perturbation analysis akin to that of Saberi and Khalil (1984) and Corless *et al.* (1989), a threshold measure μ^* of "fastness" of the compensator dynamics; to ensure overall system stability, is then derived.

The outline of the chapter is as follows. First, in § 4.2, we introduce the class of systems to be considered. In the next section, we propose a linear dynamic output feedback compensator for system introduced in § 4.2. Then, by an analogous approach, in § 4.4, we generalize the control design proposed in the previous section, to include more general systems by admitting a nonlinear discontinuous control component, modelled by an appropriately chosen set-valued map, and the overall controlled system is consequently interpreted in the generalized sense of a controlled differential inclusion (Aubin and Cellina 1984).

4.2 The system and assumptions

We consider uncertain nonlinearly perturbed linear systems of the form

$$\dot{x}(t) = Ax(t) + B[u(t) + g(t, x(t), u(t))], \quad x(t) \in \mathbb{R}^n \ , \quad u(t) \in \mathbb{R}^m \tag{4.1}$$

for which the only available state information is provided by the output

$$y(t) = Cx(t), \quad y(t) \in \mathbb{R}^p, \quad m \leq p \leq n \ . \tag{4.2}$$

The triple (C, A, B), which defines the nominal linear system, is assumed to satisfy the following.

A4.1: (A, B) is a controllable pair and B has full rank m.

A4.2: For some integer $r \geq 1$, there exist known matrices $F_1, F_2, \cdots, F_r \in \mathbb{R}^{m \times p}$, such that

(i) for $i = 1, 2, \cdots, r-1$,

$$\text{im } CA^{i-1}B \subset \bigcap_{j=i+1}^{r} \ker F_j \; ;$$

moreover, the matrix

$$C_r := F_1 C + F_2 CA + \cdots + F_r CA^{r-1}$$

is such that

(ii) $|C_r B| \neq 0$;

(iii) the transmission zeros of the m-input m-output linear system (C_r, A, B) lie in $\mathbb{C}^-$.

Example 4.1

If

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then the above assumptions hold with $r = 2$, $F_1 = [1 \;\; 1]$ and $F_2 = [1 \;\; 0]$.

Next, we impose some structure on the uncertain function g.

A4.3: g: $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function, with

(i) $\|g(t, x, u)\| \leq \alpha\|x\| + \beta\|u\|$ for all $(t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, where α and β are known constants with $\beta < 1$;

(ii) if $r \geq 2$, then g is uniformly Lipschitz in its final argument (with known Lipschitz constant λ), i.e. if $r \geq 2$, there exists known λ, such that, for each $(t,x) \in \mathbb{R}\times\mathbb{R}^n$,

$$\|g(t,x,u) - g(t,x,v)\| \leq \lambda\|u-v\| \,, \quad \text{for all } u,v \in \mathbb{R}^m \,.$$

Remark

In the terminology of Corless and Leitmann (1981), Barmish, Corless and Leitmann (1983) and Ryan and Corless (1984), the *matching condition* is implicit in (4.1).

4.3 Linear output feedback control

This section is concerned with the problem of designing a (dynamic) output feedback compensator for system (4.1,4.2). This is accomplished by initially considering system (4.1) with hypothetical output

$$y_h(t) = C_r x(t) \tag{4.3}$$

where C_r is defined as in A4.2. Note that, if $r = 1$ then $y_h(t) = F_1 y(t)$ and hence is realizable; however, if $r \geq 2$ then $y_h(t)$ is unavailable to the controller, hence the qualifier "hypothetical". For the system (4.1,4.3) so defined, (ii) and (iii) of A4.2 in essence play the role of "relative degree one" and "minimum phase" conditions on the hypothetical nominal linear system triple (C_r, A, B). Under such conditions, it is known (see, for example, Byrnes and Isidori 1984, Byrnes and Willems 1984, Mårtensson 1985 and Byrnes *et al.* 1986) that the zero state of system (4.1,4.3) can be rendered globally uniformly asymptotically stable by static output feedback; this is considered in § 4.3.1 and is reiterated in Theorem 4.1. However, with the exception of the case $r = 1$, such static output feedback is unrealizable in the context of the true system (4.1,4.2). Therefore, in § 4.3.2, a realizable dynamic compensator is constructed for the cases $r \geq 2$, which filters the actual output y. This filter can be interpreted as providing a realizable approximation to the static hypothetical output feedback; moreover, it is shown in Theorem 4.2 that global uniform asymptotic stability of the zero state of (4.1,4.2) is guaranteed provided that the filter dynamics are sufficiently fast (a calculable threshold measure of fastness is provided).

The subject of this section, can be found in Ryan and Yaacob (1989).

4.3.1 Stabilizing static output feedback for hypothetical system

For convenience, the following state transformation is introduced. Let $T_1 \in \mathbb{R}^{(n-m)\times n}$ be such that $\ker T_1 = \operatorname{im} B$, then

$$T = \begin{bmatrix} T_1 \\ (C_r B)^{-1} C_r \end{bmatrix} \text{ with inverse } T^{-1} = [S_1 \vdots B] \tag{4.4}$$

is a similarity transformation which takes system (4.1,4.3) into the form

$$\dot{\tilde{x}}(t) = A_{11}\tilde{x}(t) + A_{12}\tilde{y}(t), \quad \tilde{x}(t) \in \mathbb{R}^{n-m} \tag{4.5a}$$

$$\dot{\tilde{y}}(t) = A_{21}\tilde{x}(t) + A_{22}\tilde{y}(t) + u(t) + \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), u(t)), \quad \tilde{y}(t) \in \mathbb{R}^m \tag{4.5b}$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := TAT^{-1}, \quad \tilde{g}(t, \tilde{x}, \tilde{y}, u) := g(t, S_1\tilde{x} + B\tilde{y}, u) \tag{4.5c}$$

with hypothetical output

$$y_h(t) = (C_r B)\tilde{y}(t) \tag{4.6}$$

Note that the eigenvalues of A_{11} coincide with the transmission zeros of (C_r, A, B); thus, by virtue of A4.2(iii), $\sigma(A_{11}) \subset \mathbb{C}^-$.

Let $P > 0$ be the unique symmetric positive definite solution of the Lyapunov equation

$$PA_{11} + A_{11}^T P + I = 0 \tag{4.7}$$

then we state our first result.

Theorem 4.1

Define $\kappa^* := \|A_{22}\| + \alpha\|B\| + \frac{1}{2}[\|PA_{12} + A_{21}^T\| + \alpha\|S_1\|]^2$, then, for each fixed $\hat{\kappa} > \kappa^*(1-\beta)^{-1}$, the static output feedback

$$u(t) = -\hat{\kappa}(C_r B)^{-1} y_h(t) = -\hat{\kappa}\tilde{y}(t) \tag{4.8}$$

renders the zero state of the hypothetical system (4.1,4.3) globally uniformly asymptotically stable. •

Proof

In view of (4.7), we introduce a function V: $\mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^+$ by

$$V(\tilde{x}, \tilde{y}) := \tfrac{1}{2}\langle \tilde{x}, P\tilde{x}\rangle + \tfrac{1}{2}\|\tilde{y}\|^2 . \tag{4.9}$$

Then, along solutions $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ of (4.5,4.6,4.8) (equivalent to (4.1,4.3,4.8)), the following holds almost everywhere

$$\frac{d}{dt} V(\tilde{x}(t), \tilde{y}(t)) = -\tfrac{1}{2}\|\tilde{x}(t)\|^2 + \langle \tilde{x}(t), [PA_{12} + A_{21}^T]\tilde{y}(t)\rangle$$

$$+ \langle \tilde{y}(t), A_{22}\tilde{y}(t)\rangle$$

$$+ \langle \tilde{y}(t), -\hat{\kappa}\tilde{y}(t) + \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), -\kappa\tilde{y}(t))\rangle \tag{4.10}$$

In view of A4.3(i), (4.4) and (4.8),

$$\langle \tilde{y}(t), -\hat{\kappa}\tilde{y}(t) + \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), -\hat{\kappa}\tilde{y}(t))\rangle \leq -[\hat{\kappa}(1-\beta) - \alpha\|B\|]\|\tilde{y}(t)\|^2$$

$$+ \alpha\|S_1\|\|\tilde{x}(t)\|\|\tilde{y}(t)\| \qquad (4.11)$$

and combining (4.10) with (4.11) yields

$$\frac{d}{dt} V(\tilde{x}(t), \tilde{y}(t)) \leq -U(\tilde{x}(t), \tilde{y}(t)) \quad \text{a.e.} \qquad (4.12a)$$

where

$$U(\tilde{x}, \tilde{y}) := \frac{1}{2}\left\langle \begin{bmatrix} \|\tilde{x}\| \\ \|\tilde{y}\| \end{bmatrix}, M_{\hat{\kappa}} \begin{bmatrix} \|\tilde{x}\| \\ \|\tilde{y}\| \end{bmatrix} \right\rangle, \qquad (4.12b)$$

$$M_{\hat{\kappa}} := \begin{bmatrix} 1 & -[\|PA_{12} + A_{21}^T\| + \alpha\|S_1\|] \\ -[\|PA_{12} + A_{21}^T\| + \alpha\|S_1\|] & 2[\hat{\kappa}(1-\beta) - \|A_{22}\| - \alpha\|B\|] \end{bmatrix} \qquad (4.12c)$$

Noting that $M_{\hat{\kappa}}$ is positive definite, thus U is positive definite quadratic form, then the requisite properties of global uniform asymptotic stability may be concluded by standard arguments.

In the context of the true system (4.1,4.2), if $r = 1$, then the static output feedback (4.8) is realizable as

$$u(t) = -\hat{\kappa}(C_r B)^{-1} F_1 y(t) \qquad (4.13)$$

whence

Corollary 4.1

Let κ^* be as in Theorem 4.1. If $r = 1$ then the static output feedback (4.13) renders the zero state of the true system (4.1,4.2) globally uniformly asymptotically stable.

However, in all other cases ($r \geq 2$), the feedback (4.8) is unrealizable for the true system (4.1,4.2); in its place, we will develop a realizable dynamic compensator in the next sub-section.

4.3.2 Cases r ≥ 2: Stabilizing dynamic output feedback for the true system (4.1,4.2)

In view of A4.2(i), we note that

$$y_h(t) = C_r x(t) = F_1 y(t) + F_2 \dot{y}(t) + \cdots + F_r y^{(r-1)}(t) \tag{4.14}$$

which can be interpreted in the frequency domain as

$$\bar{y}_h(s) = [F_1 + N(s)]\bar{y}(s) , \tag{4.15a}$$

where

$$N(s) = sF_2 + s^2 F_3 + \cdots + s^{r-1} F_r \tag{4.15b}$$

is physically unrealizable. Our approach is to replace $N(s)$ in (4.15) by a physically realizable transfer matrix (filter) of the form $G_\mu(s)N(s)$ with appropriately chosen $G_\mu(s)$. To this end, let $\delta_i \leq r-1$ denote the degree of the highest-degree polynomial in the ith row of $N(s)$. Let constants $a_j^i > 0$, $j = 2, \cdots, \delta_i$, be such that

$$\chi_i(s) = s^{\delta_i} + a_{\delta_i}^i s^{\delta_i - 1} + \cdots + a_2^i s + 1 , \quad i = 1, 2, \cdots, m \tag{4.16}$$

is Hurwitz (i.e. with all its roots lying in $\mathbb{C}^-$). For $i = 1, 2, \cdots, m$, define $\Psi_i^\mu(s)$, parameterized by $\mu > 0$, as

$$\Psi_i^\mu(s) = \frac{1}{\chi_i(\mu s)} \tag{4.17}$$

which, interpreted as a transfer function, has minimal realization

$(c_i^T, \mu^{-1}A_i, \mu^{-1}b_i)$, where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -a_2^i & -a_3^i & \cdots & -a_{\delta_i}^i \end{bmatrix} \in \mathbb{R}^{\delta_i \times \delta_i}, \tag{4.18a}$$

$$b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\delta_i}, \quad c_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{\delta_i}. \tag{4.18b}$$

We now introduce the transfer matrix

$$G_\mu(s) := \text{diag}\,\{\Psi_i^\mu(s)\} \tag{4.19}$$

which clearly has minimal realization $(C^*, \mu^{-1}A^*, \mu^{-1}B^*)$, where

$$A^* = \text{diag}\,\{A_i\} \in \mathbb{R}^{q \times q}, \quad B^* = \text{diag}\,\{b_i\} \in \mathbb{R}^{q \times m}, \quad C^* = \text{diag}\,\{c_i^T\} \in \mathbb{R}^{m \times q}, \tag{4.20}$$

with $q := \sum_{i=1}^{m} \delta_i$. We note, in passing, that $\sigma(A^*) \subset \mathbb{C}^-$ and that $C^*(A^*)^{-1}B^* = -I$.

Let κ^* be as in Theorem 4.1, then, for fixed $\hat{k} > \kappa^*(1-\beta)^{-1}$, the proposed physically realizable compensator (which filters the actual output y) for system (4.1,4.2) is parameterized by μ, and has frequency domain characterization:

$$H_\mu(s) = -\hat{k}(C_r B)^{-1}[F_1 + G_\mu(s)N(s)]. \tag{4.21}$$

For notational convenience, we introduce functions φ, f_1, f_2, Δf_2, and f_3, defined as follows.

$$\varphi\colon (\tilde{x},\tilde{y},\tilde{z}) \mapsto -\hat{k}(C_rB)^{-1}\left[F_1C[S_1\tilde{x}+B\tilde{y}] + C^*\tilde{z}\right] \tag{4.22a}$$

$$f_1\colon (\tilde{x},\tilde{y}) \mapsto A_{11}\tilde{x} + A_{12}\tilde{y} \tag{4.22b}$$

$$f_2\colon (t,\tilde{x},\tilde{y}) \mapsto A_{21}\tilde{x} + A_{22}\tilde{y} - \hat{k}\tilde{y} + \tilde{g}(t,\tilde{x},\tilde{y},-\hat{k}\tilde{y}) \tag{4.22c}$$

$$\Delta f_2\colon (t,\tilde{x},\tilde{y},\tilde{z}) \mapsto \hat{k}\tilde{y} + \varphi(\tilde{x},\tilde{y},\tilde{z}) + \tilde{g}(t,\tilde{x},\tilde{y},\varphi(\tilde{x},\tilde{y},\tilde{z})) - \tilde{g}(t,\tilde{x},\tilde{y},-\hat{k}\tilde{y}) \tag{4.22d}$$

$$f_3\colon (\tilde{x},\tilde{y},\tilde{z}) \mapsto A^*\tilde{z} + B^*\left[C_rB\tilde{y} - F_1C[S_1\tilde{x}+B\tilde{y}]\right] . \tag{4.22e}$$

Then it is readily verified that, in the time domain and under state transformation T, the differential equations governing the dynamic output feedback controlled system may now be expressed in the form:

$$\dot{\tilde{x}}(t) = f_1(\tilde{x}(t),\tilde{y}(t)), \quad \tilde{x}(t) \in \mathbb{R}^{n-m} \tag{4.23a}$$

$$\dot{\tilde{y}}(t) = f_2(t,\tilde{x}(t),\tilde{y}(t)) + \Delta f_2(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)), \quad \tilde{y}(t) \in \mathbb{R}^m \tag{4.23b}$$

$$\mu\dot{\tilde{z}}(t) = f_3(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)), \quad \tilde{z}(t) \in \mathbb{R}^q . \tag{4.23c}$$

In analysing the stability of system (4.23), we regard μ as a singular perturbation parameter. Recalling that $C^*(A^*)^{-1}B^* = -I$, we note that system (4.5) with control (4.8) is recovered on setting $\mu = 0$ in (4.23); thus, in the usual terminology (Saberi and Khalil 1984, Corless *et al.* 1989 and Kokotović *et al.* 1986), system (4.5,4.8) may be interpreted as the reduced-order system associated with the singularly perturbed system (4.23). The ensuing approach is akin to that of Saberi and Khalil (1984) and Corless *et al.* (1989), our objective being to determine a threshold value $\mu^* > 0$ such that, for all $\mu \in (0,\mu^*)$, the zero state of system (4.23) is globally uniformly asymptotically stable.

Recalling that $\sigma(A^*) \subset \mathbb{C}^-$, let $P^* > 0$ be the unique symmetric positive definite solution of the Lyapunov equation

$$P^*A^* + (A^*)^T P^* + I = 0 . \tag{4.24}$$

Define W: $\mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^+$ by

$$W(\tilde{x}, \tilde{y}, \tilde{z}) := \tfrac{1}{2}\langle w(\tilde{x}, \tilde{y}, \tilde{z}),\, P^* w(\tilde{x}, \tilde{y}, \tilde{z})\rangle \tag{4.25a}$$

where

$$w(\tilde{x}, \tilde{y}, \tilde{z}) := \tilde{z} + (A^*)^{-1} B^* \left[C_r B\tilde{y} - F_1 C[S_1\tilde{x} + B\tilde{y}]\right]$$

$$= (A^*)^{-1} f_3(\tilde{x}, \tilde{y}, \tilde{z}) . \tag{4.25b}$$

We now establish some preliminary lemmas.

Lemma 4.1

$$\langle \nabla_{\tilde{x}} V(\tilde{x}, \tilde{y}),\, f_1(\tilde{x}, \tilde{y})\rangle + \langle \nabla_{\tilde{y}} V(\tilde{x}, \tilde{y}),\, f_2(t, \tilde{x}, \tilde{y})\rangle \le -\alpha_0 V(\tilde{x}, \tilde{y})$$

where

$$\alpha_0 := [\|M_{\hat{\kappa}}^{-1}\| [\|P\| + 1]]^{-1} > 0 .$$

Proof

This is implicit in the proof of Theorem 4.1. Thus, from (4.12),

$$\langle \nabla_{\tilde{x}} V(\tilde{x}, \tilde{y}),\, f_1(\tilde{x}, \tilde{y})\rangle + \langle \nabla_{\tilde{y}} V(\tilde{x}, \tilde{y}),\, f_2(t, \tilde{x}, \tilde{y})\rangle$$

$$\le -\tfrac{1}{2}\left\langle \begin{bmatrix} \|\tilde{x}\| \\ \|\tilde{y}\| \end{bmatrix},\, M_{\hat{\kappa}} \begin{bmatrix} \|\tilde{x}\| \\ \|\tilde{y}\| \end{bmatrix} \right\rangle$$

$$\le -\tfrac{1}{2} \|M_{\hat{\kappa}}^{-1}\|^{-1} \left\| \begin{matrix} \|\tilde{x}\| \\ \|\tilde{y}\| \end{matrix} \right\|^2$$

$$\le -\tfrac{1}{2} \|M_{\hat{\kappa}}^{-1}\|^{-1} [\|\tilde{x}\|^2 + \|\tilde{y}\|^2] \tag{4.26}$$

Now, V defined in (4.9) can be written as

$$V(\tilde{x}, \tilde{y}) = \frac{1}{2} \left\langle \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}, \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \right\rangle$$

Therefore

$$V(\tilde{x}, \tilde{y}) \leq \tfrac{1}{2} [\|P\| + 1] [\|\tilde{x}\|^2 + \|\tilde{y}\|^2] \tag{4.27}$$

Combining (4.26) and (4.27), the required result follows.

Lemma 4.2

$$\langle \nabla_{\tilde{z}} W(\tilde{x}, \tilde{y}, \tilde{z}), f_3(\tilde{x}, \tilde{y}, \tilde{z}) \rangle \leq -\beta_0 W(\tilde{x}, \tilde{y}, \tilde{z})$$

where

$$\beta_0 := \|P^*\|^{-1} > 0 \ .$$

Proof

$$\begin{aligned} \langle \nabla_{\tilde{z}} W(\tilde{x}, \tilde{y}, \tilde{z}), f_3(\tilde{x}, \tilde{y}, \tilde{z}) \rangle &= \langle P^* w(\tilde{x}, \tilde{y}, \tilde{z}), f_3(\tilde{x}, \tilde{y}, \tilde{z}) \rangle \\ &= \langle P^* w(\tilde{x}, \tilde{y}, \tilde{z}), A^* w(\tilde{x}, \tilde{y}, \tilde{z}) \rangle \\ &= -\tfrac{1}{2} \|w(\tilde{x}, \tilde{y}, \tilde{z})\|^2 \\ &\leq -\|P^*\|^{-1} W(\tilde{x}, \tilde{y}, \tilde{z}) \ . \end{aligned}$$

Lemma 4.3

There exists a calculable constant θ_0 such that, for all $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^q$,

$$\langle \nabla_{\tilde{x}} W(\tilde{x}, \tilde{y}, \tilde{z}), f_1(\tilde{x}, \tilde{y}) \rangle \leq \theta_0 V^{\frac{1}{2}}(\tilde{x}, \tilde{y}) W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z}) \ .$$

Proof (Sketch)

$\nabla_{\tilde{x}} W(\tilde{x}, \tilde{y}, \tilde{z}) = -[(A^*)^{-1}B^*F_1CS_1]^T P^* w(\tilde{x}, \tilde{y}, \tilde{z})$, and so, in view of (4.25b), $\|\nabla_{\tilde{x}} W(\tilde{x}, \tilde{y}, \tilde{z})\|$ is bounded above by a calculable scalar multiple of $W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z})$. Clearly, the function $\|f_1(\tilde{x}, \tilde{y})\|$ is bounded above by a calculable scalar multiple of $V^{\frac{1}{2}}(\tilde{x}, \tilde{y})$. Hence, the required result follows.

Lemma 4.4

There exist calculable constants ψ_1, ψ_2 such that, for all $(t, \tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R} \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^q$,

$$\langle \nabla_{\tilde{y}} W(\tilde{x}, \tilde{y}, \tilde{z}),\, f_2(t, \tilde{x}, \tilde{y}) + \Delta f_2(t, \tilde{x}, \tilde{y}, \tilde{z}) \rangle \le \psi_1 W(\tilde{x}, \tilde{y}, \tilde{z}) + \psi_2 V^{\frac{1}{2}}(\tilde{x}, \tilde{y}) W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z}) .$$

Proof (Sketch)

$\nabla_{\tilde{y}} W(\tilde{x}, \tilde{y}, \tilde{z}) = [(A^*)^{-1}B^*[C_rB - F_1CB]]^T P^* w(\tilde{x}, \tilde{y}, \tilde{z})$, and so, in view of (4.25b), $\|\nabla_{\tilde{y}} W(\tilde{x}, \tilde{y}, \tilde{z})\|$ is bounded above by a calculable scalar multiple of $W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z})$. In view of A4.3(i), $\|f_2(t, \tilde{x}, \tilde{y})\|$ is bounded above by a calculable scalar multiple of $V^{\frac{1}{2}}(\tilde{x}, \tilde{y})$. By A4.3(ii), $\tilde{g}$ is uniformly Lipschitz in its final argument (with known Lipschitz constant λ); hence,

$$\|\Delta f_2(t, \tilde{x}, \tilde{y}, \tilde{z})\| \le (1 + \lambda) \| \hat{K}\tilde{y} + \varphi(\tilde{x}, \tilde{y}, \tilde{z}) \|$$

for all $(t, \tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R} \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^q$, and, since $\hat{K}\tilde{y} + \varphi(\tilde{x}, \tilde{y}, \tilde{z}) = -\hat{K}(C_rB)^{-1}C^* w(\tilde{x}, \tilde{y}, \tilde{z})$ (by using $C^*(A^*)^{-1}B^* = -I$), it follows that $\|f_2(t, \tilde{x}, \tilde{y})\|$ is bounded by a calculable scalar multiple of $W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z})$. Hence, the result follows.

Lemma 4.5

There exists a calculable constant η_0 such that, for all $(t, \tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R} \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^q$,

$$\langle \nabla_{\tilde{y}} V(\tilde{x}, \tilde{y}), \Delta f_2(t, \tilde{x}, \tilde{y}, \tilde{z}) \rangle \leq \eta_0 V^{\frac{1}{2}}(\tilde{x}, \tilde{y}) W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z}) \,.$$

Proof (Sketch)

$\nabla_{\tilde{y}} V(\tilde{x}, \tilde{y}) = \tilde{y}$, and so $\|\nabla_{\tilde{y}} V(\tilde{x}, \tilde{y})\|$ is bounded above by a calculable scalar multiple of $V^{\frac{1}{2}}(\tilde{x}, \tilde{y})$. From the discussion in Lemma 4.4, $\|\Delta f_2(t, \tilde{x}, \tilde{y}, \tilde{z})\|$ is bounded above by a calculable scalar multiple of $W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z})$. Hence, the lemma follows.

Having established the above preliminary lemmas, we demonstrate in the next theorem that system (4.23) is globally uniformly asymptotically stable for all $\mu > 0$ sufficiently small.

Theorem 4.2

Let κ^* be as in Theorem 4.1 and define

$$\mu^* := \frac{\alpha_0 \beta_0}{[\alpha_0 \psi_1 + \eta_0(\theta_0 + \psi_2)]} > 0 \,.$$

Then, for each fixed $\hat{\kappa} > \kappa^*(1-\beta)^{-1}$ and fixed $\mu \in (0, \mu^*)$, the zero state of system (4.23) is globally uniformly asymptotically stable.

Proof

Define the positive definite quadratic form (Lyapunov function candidate) $\mathcal{W}$ by

$$\mathcal{W}(\tilde{x}, \tilde{y}, \tilde{z}) := V(\tilde{x}, \tilde{y}) + \eta_0(\theta_0 + \psi_2)^{-1} W(\tilde{x}, \tilde{y}, \tilde{z})$$

then, along solutions $(\tilde{x}(\cdot),\tilde{y}(\cdot),\tilde{z}(\cdot))$ of (4.23), the following holds almost everywhere

$$\frac{d}{dt}\mathcal{W}(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) = \langle \nabla_{\tilde{x}} V(\tilde{x}(t),\tilde{y}(t)),\ f_1(\tilde{x}(t),\tilde{y}(t))\rangle$$

$$+ \langle \nabla_{\tilde{y}} V(\tilde{x}(t),\tilde{y}(t)),\ f_2(t,\tilde{x}(t),\tilde{y}(t))\rangle$$

$$+ \langle \nabla_{\tilde{y}} V(\tilde{x}(t),\tilde{y}(t)),\ \Delta f_2(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\rangle$$

$$+ k_0 \big[\langle \nabla_{\tilde{x}} W(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)),\ f_1(\tilde{x}(t),\tilde{y}(t))\rangle$$

$$+ \langle \nabla_{\tilde{y}} W(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)),\ f_2(t,\tilde{x}(t),\tilde{y}(t))$$

$$+ \Delta f_2(t,\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\rangle$$

$$+ \mu^{-1}\langle \nabla_{\tilde{z}} W(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)),\ f_3(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\rangle\big]$$

where $k_0 := \eta_0(\theta_0+\psi_2)^{-1}$. By invoking Lemmas 4.1-4.5, the following holds almost everywhere along solutions $(\tilde{x}(\cdot),\tilde{y}(\cdot),\tilde{z}(\cdot))$ of (4.23),

$$\frac{d}{dt}\mathcal{W}(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \le -\left\langle \begin{bmatrix} V^{\frac{1}{2}}(\tilde{x}(t),\tilde{y}(t)) \\ W^{\frac{1}{2}}(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \end{bmatrix},\ M_\mu \begin{bmatrix} V^{\frac{1}{2}}(\tilde{x}(t),\tilde{y}(t)) \\ W^{\frac{1}{2}}(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \end{bmatrix}\right\rangle \tag{4.28a}$$

where

$$M_\mu := \begin{bmatrix} \alpha_0 & -\eta_0 \\ -\eta_0 & (\mu^{-1}\beta_0-\psi_1)(\theta_0+\psi_2)^{-1}\eta_0 \end{bmatrix}. \tag{4.28b}$$

Noting that M_μ is positive definite; hence, the result follows.

4.4 Discontinuous output feedback control

The aim of this section is to extend the approach proposed in § 4.3, by generalizing the class of allowable uncertainties. A generalized output feedback control is developed which renders the zero state globally uniformly asymptotically stable. The generalized feedback has a linear plus discontinuous output feedback structure. The discontinuous control component is modelled by an appropriately chosen set-valued map, and we adopt the analytic framework of controlled differential inclusions (Aubin and Cellina 1984).

The approach adopted here is essentially that of Ryan (1988) and Leitmann and Ryan (1987). In Ryan (1988) and for the case $r = 1$ only, a wider class of uncertain functions g is studied. Specifically, he has considered a class of nonlinear systems with uncertain functions g satisfying

$$\|g(t,x,u)\| \leq \alpha\|x\| + \beta\|u\| + \gamma\xi(Cx) \tag{4.29}$$

for all $(t,x,u) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$ with α and $\beta < 1$ as in A4.3 and where γ is a known constant and ξ is a known continuous function. Thus, in Ryan (1988) a non-cone-bounded component of uncertainty is allowed but this is required to be bounded by a function of the system output y. Here, we will consider the cases $r \geq 2$, by using an approach of Leitmann and Ryan (1987) on decomposition of the uncertain function g. Thus, the subject consider here may be regarded as an extension of § 4.3 and Ryan (1988); however, this extension is achieved at the expense of additional assumptions on the "hypothetical" nominal system and on the uncertain function g, which are stated in the following sub-section.

The approach used in the present section is analogous to that described in § 4.3, but, in contrast to § 4.3, a discontinuous control component is admitted and the overall controlled system is consequently interpreted in the generalized sense of a controlled differential inclusion (Aubin and Cellina 1984). Thus, in § 4.4.2, we consider a hypothetical output y_h defined as in (4.3) for system (4.1) and establish the existence of a stabilizing generalized static output feedback for the hypothetical system; this is stated in Theorem 4.3. Since this generalized static output feedback is unrealizable in the context of the true system (4.1,4.2) (except for the case $r = 1$), in § 4.4.3, we will construct a realizable generalized dynamic compensator for the cases $r \geq 2$, which filters the actual output y. As we have mentioned in § 4.3, this filter can be interpreted as providing a realizable approximation to the generalized static hypothetical output feedback; furthermore, it will be shown in Theorem 4.4 that global uniform asymptotic stability of the zero state of (4.1,4.2) is guaranteed provided that the filter dynamics are sufficiently fast.

4.4.1 Additional assumptions

Consider again system (4.1,4.2). Here however, we have to impose some additional conditions on the system. Before that, we need the following.

Let Π denote the matrix of orthogonal projection of $\mathbb{R}^m$ onto

$$\mathcal{S} = (\text{im}\,[(C_r B)^{-1}(\sum_{j=1}^{r-1} F_{j+1} C A^j)])^{\perp} \subset \mathbb{R}^m \tag{4.30}$$

In the next assumption, additional structural properties are imposed on the uncertain function g. In particular, we have to replace A4.3(i), however, A4.3(ii) remains in force. Thus, A4.3(i) is now replaced by:

A4.4: There exist known non-negative constants α, β_1, β_2, γ, a Carathéodory function $g_1\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$, a continuous function $g_2\colon \mathbb{R}^m \to \mathbb{R}^m$, and a known continuous function $\xi\colon \mathbb{R}^p \to \mathbb{R}^+$ such that, for all $(t,x,u) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$,

(i) $g(t,x,u) = g_1(t,x) + g_2(u)$;

(ii) $\|(I-\Pi)g_1(t,x)\| \le \alpha\|x\|$;

(iii) $\|\Pi g_1(t,x)\| \le \gamma\xi(Cx)$;

(iv) $\|(I-\Pi)g_2(u)\| \le \beta_1\|(I-\Pi)u\|$, $\beta_1 < 1$;

(v) $\|\Pi g_2(u)\| \le \beta_2\|\Pi u\|$, $\beta_2 < 1$.

Example 4.2

If

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then the assumptions A4.1 and A4.2 hold with $r = 2$,

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, $\mathcal{S}$ defined as in (4.30) is given by

$$\mathcal{S} = \left[\operatorname{im}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]^{\perp} = \{(u_1, u_2) \mid u_1 = 0\}$$

with

$$\Pi = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We wish to admit discontinuous control. Clearly, if such discontinuous control is coupled with system (4.1), the resulting system is a differential equation with discontinuous right hand side. For such equations, the classical (Carathéodory) theory and concept of solution are inappropriate; consequently, the discontinuous feedback system is interpreted in the sense of generalized dynamical system (Gutman 1979, Leitmann 1979) and defined via a differential inclusion (Aubin and Cellina 1984, Clarke 1983). Now, we are going to recast the problem in the context of controlled differential inclusions.

From A4.4, we first have the following.

Proposition 4.1

For each function g_1 satisfying A4.4(ii)-(iii),

$$g_1(t,x) \in \mathrm{G}_1(x) := (I-\Pi)\bar{\mathbb{B}}_m(\alpha\|x\|) + \Pi\bar{\mathbb{B}}_m(\gamma\xi(Cx)) \subset \mathbb{R}^m$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$.

Proof

Let g_1 satisfy A4.4(ii)-(iii). Then

$$g_1(t,x) = (I-\Pi)g_1(t,x) + \Pi g_1(t,x)$$

$$= v_1 + v_2$$

with $\|v_1\| \le \alpha\|x\|$ and $\|v_2\| \le \gamma\xi(Cx)$. Hence,

$$v_1 \in (I-\Pi)\bar{\mathbb{B}}_m(\alpha\|x\|) \quad \text{and} \quad v_2 \in \Pi\bar{\mathbb{B}}_m(\gamma\xi(Cx)) \ ,$$

which completes the proof.

Now, system (4.1) with output feedback (4.2) is replaced by the differential inclusion system

$$\dot{x}(t) \in Ax(t) + B[u(t) + \mathbf{G}_1(x(t)) + g_2(u(t))] \tag{4.31}$$

with output

$$y(t) = Cx(t) \tag{4.32}$$

Certainly, for each bounded measurable function $u(\cdot)$, any solution $x(\cdot)$ of (4.1) (absolutely continuous function satisfying (4.1) a.e.) is also a solution of (4.31) (absolutely continuous function satisfying (4.31) a.e.).

It is clearly seen that, $\mathbf{G}_1$ defined as in Proposition 4.1 has convex and compact values. Moreover, since ξ is continuous then $\mathbf{G}_1$ is upper semi-continuous (in fact continuous).

Our first task now is to establish the existence of a generalized output feedback $(y, z) \mapsto \mathbf{H}_1(y, z)$, which renders the zero state of the feedback controlled differential inclusion

$$\dot{x}(t) \in \mathbf{F}(x(t)) \tag{4.33a}$$

where

$$\mathbf{F}(x) := Ax + B\left[\mathbf{H}_1(Cx, (C_rB)^{-1}C_rx) + \mathbf{G}_1(x) + \mathbf{G}_{21}(x)\right] \tag{4.33b}$$

$$\mathbf{G}_{21}(x) := \{g_2(u) \colon u \in \mathbf{H}_1(Cx, (C_rB)^{-1}C_rx)\} \tag{4.33c}$$

globally uniformly asymptotically stable.

4.4.2 Existence of stabilizing generalized static output feedback for hypothetical system

By using a similarity transformation as introduced in § 4.3.1, then under transformation T as defined in (4.4) takes system (4.1,4.3) into the form

$$\dot{\tilde{x}}(t) = A_{11}\tilde{x}(t) + A_{12}\tilde{y}(t), \quad \tilde{x}(t) \in \mathbb{R}^{n-m} \tag{4.34a}$$

$$\dot{\tilde{y}}(t) \in A_{21}\tilde{x}(t) + A_{22}\tilde{y}(t) + u(t) + \tilde{\mathbf{G}}_1(\tilde{x}(t), \tilde{y}(t)) + g_2(u(t)), \ \tilde{y}(t) \in \mathbb{R}^m \tag{4.34b}$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := TAT^{-1} \ ; \ \tilde{\mathbf{G}}_1(\tilde{x}, \tilde{y}) := \mathbf{G}_1(T^{-1}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}) \tag{4.34c}$$

with hypothetical output

$$y_h(t) = (C_r B)\tilde{y}(t) \tag{4.35}$$

Recalling that the eigenvalues of A_{11} coincide with the transmission zeros of (C_r, A, B); thus, by virtue of A4.2(iii), $\sigma(A_{11}) \subset \mathbb{C}^-$. Hence, the Lyapunov equation (4.7) has a unique symmetric positive definite solution $P > 0$. Define the matrix $M_{\hat{\kappa}_d}$ by

$$M_{\hat{\kappa}_d} := \begin{bmatrix} 1 & -(m_1+m_3) & -m_1 \\ -(m_1+m_3) & 2[\hat{\kappa}_d(1-\beta_1)-m_2-m_4] & -m_4 \\ -m_1 & -m_4 & 2[\hat{\kappa}_d(1-\beta_2)-m_2] \end{bmatrix} \tag{4.36}$$

with

$$m_1 = \|PA_{12} + A_{21}^T\|, \ m_2 = \|A_{22}\|, \ m_3 = \alpha\|S_1\|, \ m_4 = \alpha\|B\|.$$

Let $\mathbf{H}_1$ be the generalized feedback given by

$$\mathbf{H}_1(y, \tilde{y}) := -\hat{\kappa}_d[\tilde{y} + \mathbf{N}(y)] \tag{4.37a}$$

where the set-valued map $y \mapsto \mathrm{N}(y) \subset \mathbb{R}^m$ in essence models a discontinuous control component and is given by

$$\mathrm{N}(y) := \begin{cases} \{\xi(y)\|\Pi(C_rB)^{-1}F_1y\|^{-1}\Pi(C_rB)^{-1}F_1y\}\,, & \Pi(C_rB)^{-1}F_1y \neq 0 \\ \overline{\mathbb{B}}_m(\xi(y))\,, & \Pi(C_rB)^{-1}F_1y = 0 \end{cases} \tag{4.37b}$$

Then we state the following.

Theorem 4.3

Define $\kappa_d^* := \inf\{\hat{\kappa}_d\colon M_{\hat{\kappa}_d} > 0\}$. Then, for each fixed $\hat{\kappa}_d > \max\{\kappa_d^*, (1-\beta_2)^{-1}\gamma\}$, the generalized static output feedback $\mathbf{H}_1$ defined in (4.37) renders the zero state of the hypothetical system (4.34,4.35) globally uniformly asymptotically stable.

Proof

Note initially that $\mathbf{H}_1$ defined in (4.37) is singleton-valued off the subspace $\Sigma_{\mathrm{H}_1} = \ker \Pi(C_rB)^{-1}F_1 \subset \mathbb{R}^p$ and is upper semi-continuous with convex and compact values; thus, $\mathbf{H}_1$ qualifies as a generalized feedback. Now, consider the transformed system (4.34) under feedback control (4.37), viz.

$$(\dot{\tilde{x}}(t), \dot{\tilde{y}}(t)) \in \mathbf{F}_1(\tilde{x}(t), \tilde{y}(t)) \tag{4.38a}$$

where

$$\mathbf{F}_1(\tilde{x}, \tilde{y}) := \{A_{11}\tilde{x} + A_{12}\tilde{y}\} \times \mathbf{D}_1(\tilde{x}, \tilde{y}) \subset \mathbb{R}^{n-m} \times \mathbb{R}^m \tag{4.38b}$$

with

$$\mathbf{D}_1(\tilde{x}, \tilde{y}) := A_{21}\tilde{x} + A_{22}\tilde{y} + \tilde{\mathbf{H}}_1(\tilde{x}, \tilde{y}) + \tilde{\mathbf{G}}_1(\tilde{x}, \tilde{y}) + \tilde{\mathbf{G}}_{21}(\tilde{x}, \tilde{y}) \tag{4.38c}$$

$$\tilde{H}_1(\tilde{x},\tilde{y}) := H_1(CT^{-1}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix},\tilde{y}) = -\hat{k}_d[\tilde{y} + \tilde{N}(\tilde{x},\tilde{y})] \tag{4.38d}$$

$$\tilde{N}(\tilde{x},\tilde{y}) := N(CT^{-1}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}) \tag{4.38e}$$

$$\tilde{G}_{21}(\tilde{x},\tilde{y}) := \{g_2(u): u \in \tilde{H}_1(\tilde{x},\tilde{y})\} . \tag{4.38f}$$

Clearly, the multifunction F_1 is upper semi-continuous with convex and compact values. Hence, for each pair $(\tilde{x}(t_0),\tilde{y}(t_0)) \in \mathbb{R}^{n-m}\times\mathbb{R}^m$, there exists a local solution $(\tilde{x},\tilde{y})$: $[t_0,\tau) \to \mathbb{R}^{n-m}\times\mathbb{R}^m$ to the above system (see Aubin and Cellina 1984).

By considering a Lyapunov function candidate V defined as in (4.9), then along every local solution $(\tilde{x}(\cdot),\tilde{y}(\cdot))$ of (4.38), the following holds almost everywhere

$$\frac{d}{dt}V(\tilde{x}(t),\tilde{y}(t)) \in -\tfrac{1}{2}\|\tilde{x}(t)\|^2 + \langle\tilde{x}(t),[PA_{12}+A_{21}^T]\tilde{y}(t)\rangle$$

$$+ \langle\tilde{y}(t),A_{22}\tilde{y}(t)\rangle + \mathcal{G}(\tilde{x}(t),\tilde{y}(t))$$

with

$$\mathcal{G}(\tilde{x},\tilde{y}) := \{\langle\tilde{y},u_1+w_1+w_2\rangle: u_1 \in \tilde{H}_1(\tilde{x},\tilde{y});\ w_1 \in \tilde{G}_1(\tilde{x},\tilde{y});\ w_2 \in \tilde{G}_{21}(\tilde{x},\tilde{y})\}$$

Now, in view of (4.30),

$$\Pi\tilde{y}(t) = \Pi(C_rB)^{-1}F_1y(t) = \Pi(C_rB)^{-1}F_1CT^{-1}[S_1\tilde{x}(t)+B\tilde{y}(t)] \tag{4.39}$$

Defining

$$\tilde{\xi}(\tilde{x},\tilde{y}) := \xi(CT^{-1}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}) \tag{4.40}$$

then, in view of definition of N and by using $\tilde{y}(t) = (I-\Pi)\tilde{y}(t) + \Pi\tilde{y}(t)$ and (4.39), for all $v \in \tilde{N}(\tilde{x}(t),\tilde{y}(t))$,

$$\langle \tilde{y}(t), v \rangle = \tilde{\xi}(\tilde{x}(t), \tilde{y}(t)) \|\Pi \tilde{y}(t)\| \tag{4.41}$$

By direct calculation,

$$\begin{aligned}
\sup \mathcal{G}(\tilde{x}, \tilde{y}) \leq & - \left[\hat{k}_d(1-\beta_1) - \alpha\|B\|\right] \|(I-\Pi)\tilde{y}\|^2 - \hat{k}_d(1-\beta_2)\|\Pi\tilde{y}\|^2 \\
& + \alpha\|S_1\| \|(I-\Pi)\tilde{y}\| \|\tilde{x}\| + \alpha\|B\| \|(I-\Pi)\tilde{y}\| \|\Pi\tilde{y}\| \\
& - \left[\hat{k}_d(1-\beta_2) - \gamma\right] \|\Pi\tilde{y}\| \tilde{\xi}(\tilde{x}, \tilde{y}) \\
\leq & - \left[\hat{k}_d(1-\beta_1) - \alpha\|B\|\right] \|(I-\Pi)\tilde{y}\|^2 - \hat{k}_d(1-\beta_2)\|\Pi\tilde{y}\|^2 \\
& + \alpha\|S_1\| \|(I-\Pi)\tilde{y}\| \|\tilde{x}\| + \alpha\|B\| \|(I-\Pi)\tilde{y}\| \|\Pi\tilde{y}\|
\end{aligned}$$

Hence,

$$\frac{d}{dt} V(\tilde{x}(t), \tilde{y}(t)) \leq - \mathcal{U}(\tilde{x}(t), \tilde{y}(t)) \quad \text{a.e.} \tag{4.42a}$$

where

$$\mathcal{U}(\tilde{x}, \tilde{y}) := \frac{1}{2} \left\langle \begin{bmatrix} \|\tilde{x}\| \\ \|(I-\Pi)\tilde{y}\| \\ \|\Pi\tilde{y}\| \end{bmatrix}, M_{\hat{k}_d} \begin{bmatrix} \|\tilde{x}\| \\ \|(I-\Pi)\tilde{y}\| \\ \|\Pi\tilde{y}\| \end{bmatrix} \right\rangle, \tag{4.42b}$$

and $M_{\hat{k}_d}$ is defined as in (4.36). Noting that $M_{\hat{k}_d}$ is a positive definite matrix and thus $\mathcal{U}$ is a positive definite quadratic form; hence the result follows.

The generalized static output feedback (4.37) is unrealizable for the true system (4.1,4.2) except for case $r = 1$. Thus, in this case ($r = 1$), the generalized static output feedback (4.37) is realizable as

$$u(t) \in - \hat{k}_d \left[(C_r B)^{-1} F_1 y(t) + \mathrm{N}(y)\right] \tag{4.43}$$

whence:

Corollary 4.2

Let κ_d^* be as in Theorem 4.3. If $r = 1$ then the generalized static output feedback (4.43) renders the zero state of the true system (4.1,4.2) globally uniformly asymptotically stable.

For all other cases ($r \geq 2$), in the next sub-section we will develop a realizable dynamic compensator which filters the actual output y. This filter can be interpreted as a realizable approximation to the generalized static hypothetical output feedback (4.37).

4.4.3 Cases r ≥ 2: Stabilizing generalized dynamic output feedback for the true system (4.1,4.2)

Recalling from the earlier part of § 4.3.2 that

$$y_h(t) = C_r x(t) = F_1 y(t) + F_2 \dot{y}(t) + \cdots + F_r y^{(r-1)}(t)$$

which can be interpreted in the frequency domain as

$$\bar{y}_h(s) = [F_1 + N(s)]\bar{y}(s) \ ,$$

where

$$N(s) = sF_2 + s^2 F_3 + \cdots + s^{r-1} F_r$$

is physically unrealizable. Our approach is to replace $N(s)$ by a physically realizable transfer matrix (filter) of the form $G_{\mu_d}(s)N(s)$ with appropriately chosen G_{μ_d}. We proceed exactly as described in § 4.3.2, so here we just briefly mentioned the procedure used.

Recalling from § 4.3.2 that we have chosen $G_\mu(s)$ as

$$G_\mu(s) := \text{diag } \{\Psi_i^\mu\}$$

where $\Psi_i^\mu(s)$ (parameterized by $\mu > 0$) is defined as in (4.17) which, interpreted as a transfer function, has minimal realization $(c_i^T, \mu^{-1}A_i, \mu^{-1}b_i)$, where A_i, b_i and c_i are given by (4.18); and $G_\mu(s)$ has minimal realization $(C^*, \mu^{-1}A^*, \mu^{-1}B^*)$, where A^*, B^* and C^* are given by (4.20). Moreover, we note that $\sigma(A^*) \subset \mathbb{C}^-$ and that $C^*(A^*)^{-1}B^* = -I$.

Let κ_d^* be as in Theorem 4.3, then, for fixed $\hat{\kappa}_d > \max\{\kappa_d^*, (1-\beta_2)^{-1}\gamma\}$ the proposed physically realizable filter (which filters the actual output y and forms the linear component of the overall compensator) for system (4.1,4.2) is parameterized by μ_d, and has transfer function,

$$H_{\mu_d}(s) = -\hat{\kappa}_d(C_rB)^{-1}[F_1 + G_{\mu_d}(s)N(s)] \tag{4.44}$$

where we have chosen $G_{\mu_d}(s) = G_\mu(s)$, while the discontinuous component is realizable and modelled by set-valued map $\mathbf{N}$ defined by (4.37b).

For notational convenience we introduce multifunctions $\mathbf{H}_2$, $\mathbf{G}_{22}$ and $\mathbf{D}_2$ as follows.

$$\mathbf{H}_2(y, \tilde{z}) := -\hat{\kappa}_d[(C_rB)^{-1}(F_1y + C^*\tilde{z}) + \mathbf{N}(y)] \tag{4.45}$$

$$\mathbf{G}_{22}(y, \tilde{z}) := \{g_2(u) \colon u \in \mathbf{H}_2(y, \tilde{z})\} \tag{4.46}$$

$$\mathbf{D}_2(\tilde{x}, \tilde{y}, \tilde{z}) := A_{21}\tilde{x} + A_{22}\tilde{y} + \tilde{\mathbf{H}}_2(\tilde{x}, \tilde{y}, \tilde{z}) + \tilde{\mathbf{H}}_1(\tilde{x}, \tilde{y}) + \tilde{\mathbf{G}}_{22}(\tilde{x}, \tilde{y}, \tilde{z}) \tag{4.47a}$$

where

$$\tilde{\mathbf{H}}_2(\tilde{x}, \tilde{y}, \tilde{z}) := \mathbf{H}_2(CT^{-1}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}, \tilde{z})$$

$$= -\hat{\kappa}_d[(C_rB)^{-1}(F_1C[S_1\tilde{x}+B\tilde{y}] + C^*\tilde{z}) + \tilde{\mathbf{N}}(\tilde{x}, \tilde{y})] \tag{4.47b}$$

$$\tilde{G}_{22}(\tilde{x},\tilde{y},\tilde{z}) := G_{22}(T^{-1}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix},\tilde{z}) = \{g_2(u):\ u \in \tilde{H}_2(\tilde{x},\tilde{y},\tilde{z})\} \quad (4.47c)$$

and $\tilde{N}$ is defined as in (4.38e).

The next proposition shows that, there is a relationship between $\tilde{H}_1$ and $\tilde{H}_2$ and between $\tilde{G}_{21}$ and $\tilde{G}_{22}$.

Proposition 4.2

For all $(\tilde{x},\tilde{y},\tilde{z}) \in \mathbb{R}^{n-m}\times\mathbb{R}^m\times\mathbb{R}^q$,

(i) $\tilde{H}_2(\tilde{x},\tilde{y},\tilde{z}) = u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) - u_{l_1}(\tilde{y}) + \tilde{H}_1(\tilde{x},\tilde{y})$;

(ii) $\tilde{G}_{22}(\tilde{x},\tilde{y},\tilde{z}) \subset \tilde{G}_{21}(\tilde{x},\tilde{y}) + \bar{\mathbb{B}}_m(\lambda\|u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) - u_{l_1}(\tilde{y})\|)$,

where

$$u_{l_1}(\tilde{y}) := -\hat{K}_d\tilde{y} \quad (4.48a)$$

$$u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) := -\hat{K}_d(C_rB)^{-1}[F_1C[S_1\tilde{x}+B\tilde{y}] + C^*\tilde{z}] \quad (4.48b)$$

Proof

(i) Let $u_2 \in \tilde{H}_2(\tilde{x},\tilde{y},\tilde{z})$. Then, from (4.48b),

$$u_2 = u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) + v,\quad v \in \tilde{N}(\tilde{x},\tilde{y})$$

$$= u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) - u_{l_1}(\tilde{y}) + u_{l_1}(\tilde{y}) + v,$$

$$= u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) - u_{l_1}(\tilde{y}) + u_1,\quad u_1 \in \tilde{H}_1(\tilde{x},\tilde{y})$$

Therefore

$$\tilde{H}_2(\tilde{x},\tilde{y},\tilde{z}) \subset \tilde{H}_1(\tilde{x},\tilde{y}) + u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) - u_{l_1}(\tilde{y})$$

Now, let $u \in \tilde{H}_1(\tilde{x},\tilde{y}) + u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) - u_{l_1}(\tilde{y})$. Then,

$$u = u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) - u_{l_1}(\tilde{y}) + u_1, \quad u_1 \in \tilde{\mathbf{H}}_1(\tilde{x}, \tilde{y})$$

$$= u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) - u_{l_1}(\tilde{y}) + u_{l_1}(\tilde{y}) + v, \quad v \in \tilde{\mathbf{N}}(\tilde{x}, \tilde{y})$$

$$= u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) + v, \quad v \in \tilde{\mathbf{N}}(\tilde{x}, \tilde{y})$$

Therefore

$$u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) - u_{l_1}(\tilde{y}) + \tilde{\mathbf{H}}_1(\tilde{x}, \tilde{y}) \subset \tilde{\mathbf{H}}_2(\tilde{x}, \tilde{y}, \tilde{z})$$

Hence, the result follows.

(ii) Let $w_2 \in \tilde{\mathbf{G}}_{22}(\tilde{x}, \tilde{y}, \tilde{z})$, then

$$w_2 = g_2(u_2), \quad u_2 = u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) + v, \quad v \in \tilde{\mathbf{N}}(\tilde{x}, \tilde{y})$$

Let $u_1 = u_{l_1}(\tilde{y}) + v$, then $u_1 \in \tilde{\mathbf{H}}_1(\tilde{x}, \tilde{y})$. Now,

$$w_2 = g_2(u_1) + g_2(u_2) - g_2(u_1)$$

$$= w_1 + g_2(u_2) - g_2(u_1), \quad w_1 \in \tilde{\mathbf{G}}_{21}(\tilde{x}, \tilde{y})$$

Then, by Lipschitz condition A4.3(ii), we have

$$\|g_2(u_2) - g_2(u_1)\| \leq \lambda \|u_2 - u_1\|$$

$$= \lambda \|u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) - u_{l_1}(\tilde{y})\|,$$

which proves the assertion (ii).

By using Proposition 4.2, we may replace $\mathbf{D}_2$ defined in (4.47a) by $\mathbf{D}_3$ where $\mathbf{D}_3 \supset \mathbf{D}_2$ and

$$\mathbf{D}_3(\tilde{x}, \tilde{y}, \tilde{z}) := \mathbf{D}_1(\tilde{x}, \tilde{y}) + u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) - u_{l_1}(\tilde{y})$$

$$+ \overline{\mathbb{B}}_m(\lambda \|u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) - u_{l_1}(\tilde{y})\|) \tag{4.49}$$

Then it can be shown that, in the time domain and under state transformation T, the differential inclusions governing the dynamic output feedback controlled system may now be put in the form

$$(\dot{\tilde{x}}(t), \dot{\tilde{y}}(t), \mu_d \dot{\tilde{z}}(t)) \in \mathrm{F}_2(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)), \quad \mu_d > 0 \tag{4.50a}$$

where

$$\mathrm{F}_2(\tilde{x}, \tilde{y}, \tilde{z}) := \{f_1(\tilde{x}, \tilde{y})\} \times \mathrm{D}_3(\tilde{x}, \tilde{y}, \tilde{z}) \times \{f_3(\tilde{x}, \tilde{y}, \tilde{z})\} \subset \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^q \tag{4.50b}$$

with real-valued functions f_1 and f_3 defined as (4.22b) and (4.22e) respectively, i.e.

$$f_1 : (\tilde{x}, \tilde{y}) \mapsto A_{11}\tilde{x} + A_{12}\tilde{y} \tag{4.50c}$$

$$f_3 : (\tilde{x}, \tilde{y}, \tilde{z}) \mapsto A^*\tilde{z} + B^*\left[C_r B\tilde{y} - F_1 C[S_1\tilde{x} + B\tilde{y}]\right] \tag{4.50d}$$

In analysing the stability of (4.50), we regard μ_d as a singular perturbation parameter. Note that system (4.34) with control (4.37) is recovered on setting $\mu_d = 0$ in (4.50); thus, in the terminology (Saberi and Khalil 1984, Corless *et al.* 1989 and Kokotović *et al.* 1986) system (4.34,4.37) may be interpreted as the reduced-order system associated with the singularly perturbed system (4.50). The ensuing approach is akin to that of Saberi and Khalil (1984) and Corless *et al.* (1989), our goal being to determine a threshold value $\mu_d^* > 0$ such that, for all $\mu_d \in (0, \mu_d^*)$, the zero state of system (4.50) is globally uniformly asymptotically stable.

Recalling again that $\sigma(A^*) \subset \mathbb{C}^-$, thus the Lyapunov equation (4.24) has a unique symmetric positive definite solution $P^* > 0$. Consider again the Lyapunov function candidate W defined as in (4.25).

Before proceeding, we impose our final assumption.

A4.5: $B^*[C_rB - F_1CB]\Pi = 0$, where Π is the matrix of orthogonal projection of $\mathbb{R}^m$ onto $\mathcal{S}$ as defined in (4.30).

We now state some preliminary lemmas (analogous to Lemmas 4.1,4.4-4.5).

Lemma 4.6

$$\langle \nabla_{\tilde{x}} V(\tilde{x},\tilde{y}),\, f_1(\tilde{x},\tilde{y})\rangle + \sup \mathcal{G}_1(\tilde{x},\tilde{y}) \le -\alpha_1 V(\tilde{x},\tilde{y})$$

where

$$\mathcal{G}_1(\tilde{x},\tilde{y}) := \{\langle \nabla_{\tilde{y}} V(\tilde{x},\tilde{y}),\, h_1\rangle : h_1 \in \mathbf{D}_1(\tilde{x},\tilde{y})\}$$

and

$$\alpha_1 := [\|M_{\hat{K}_d}^{-1}\|[\|P\|+1]]^{-1} > 0 \,.$$

Proof

The proof of this lemma is implicit in the proof of Theorem 4.3. Thus,

$$\langle \nabla_{\tilde{x}} V(\tilde{x},\tilde{y}),\, f_1(\tilde{x},\tilde{y})\rangle + \sup \mathcal{G}_1(\tilde{x},\tilde{y})$$

$$\le -\frac{1}{2}\left\langle \begin{bmatrix} \|\tilde{x}\| \\ \|(I-\Pi)\tilde{y}\| \\ \|\Pi\tilde{y}\| \end{bmatrix},\, M_{\hat{K}_d} \begin{bmatrix} \|\tilde{x}\| \\ \|(I-\Pi)\tilde{y}\| \\ \|\Pi\tilde{y}\| \end{bmatrix} \right\rangle$$

$$\le -\tfrac{1}{2}\|M_{\hat{K}_d}^{-1}\|^{-1} \left\| \begin{bmatrix} \|\tilde{x}\| \\ \|(I-\Pi)\tilde{y}\| \\ \|\Pi\tilde{y}\| \end{bmatrix} \right\|^2$$

$$= -\tfrac{1}{2}\|M_{\hat{K}_d}^{-1}\|^{-1}\, [\|\tilde{x}\|^2 + \|(I-\Pi)\tilde{y}\|^2 + \|\Pi\tilde{y}\|^2]$$

$$= -\tfrac{1}{2}\|M_{\hat{K}_d}^{-1}\|^{-1}\,[\|\tilde{x}\|^2 + \|\tilde{y}\|^2] \qquad (4.51)$$

Now, V defined in (4.7) may be written as

$$V(\tilde{x},\tilde{y}) = \frac{1}{2}\left\langle \begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}, \begin{bmatrix}P & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}\right\rangle$$

Therefore

$$V(\tilde{x},\tilde{y}) \le \tfrac{1}{2}[\|P\| + 1]\,[\|\tilde{x}\|^2 + \|\tilde{y}\|^2] \qquad (4.52)$$

Combining (4.51) and (4.52), we have the required result.

Lemma 4.7

There exist calculable constants ψ_3, ψ_4 such that, for all $(\tilde{x},\tilde{y},\tilde{z}) \in \mathbb{R}^{n-m}\times\mathbb{R}^m\times\mathbb{R}^q$,

$$\sup\, \mathcal{G}_2(\tilde{x},\tilde{y},\tilde{z}) \le \psi_3 W(\tilde{x},\tilde{y},\tilde{z}) + \psi_4 V^{\frac{1}{2}}(\tilde{x},\tilde{y})W^{\frac{1}{2}}(\tilde{x},\tilde{y},\tilde{z})$$

where

$$\mathcal{G}_2(\tilde{x},\tilde{y},\tilde{z}) := \{\langle \nabla_{\tilde{y}} W(\tilde{x},\tilde{y},\tilde{z}),\, h_3\rangle :\, h_3 \in \mathbf{D}_3(\tilde{x},\tilde{y},\tilde{z})\}$$

Proof (Sketch)

$$\nabla_{\tilde{y}} W(\tilde{x},\tilde{y},\tilde{z}) = [(A^*)^{-1}B^*[C_rB - F_1CB]]^T P^* w(\tilde{x},\tilde{y},\tilde{z})$$

$$= (M^*)^T P^* w(\tilde{x},\tilde{y},\tilde{z})$$

where

$$M^* := (A^*)^{-1}B^*[C_rB - F_1CB]\ .$$

By recalling that $\mathbf{D}_3$ as defined in (4.49), we may write $\mathcal{G}_2$ as

$$\mathcal{G}_2(\tilde{x},\tilde{y},\tilde{z}) = \{\langle \nabla_{\tilde{y}} W(\tilde{x},\tilde{y},\tilde{z}),\ h_1 + u + h\rangle :\ h_1 \in \mathbf{D}_1(\tilde{x},\tilde{y});$$

$$h \in \overline{\mathbb{B}}_m(\lambda \|u(\tilde{x},\tilde{y},\tilde{z})\|);\ u = u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) - u_{l_1}(\tilde{y})\}$$

From the definitions of $\mathbf{D}_1$, $\tilde{\tilde{\mathbf{H}}}_1$, $\tilde{\mathbf{G}}_1$ and $\tilde{\mathbf{G}}_{21}$, and making use of A4.5 (i.e. the inner product of $(M^*)^T P^* w(\tilde{x},\tilde{y},\tilde{z})$ with any terms containing "Π" is zero) and noting that $\|(M^*)^T P^* w(\tilde{x},\tilde{y},\tilde{z})\|$ is bounded above by a scalar multiple of $W^{\frac{1}{2}}(\tilde{x},\tilde{y},\tilde{z})$, we may conclude that there exists a calculable constant k_2 such that,

$$\sup\ \{\langle \nabla_{\tilde{y}} W(\tilde{x},\tilde{y},\tilde{z}),\ h_1\rangle :\ h_1 \in \mathbf{D}_1(\tilde{x},\tilde{y})\} \le k_2 V^{\frac{1}{2}}(\tilde{x},\tilde{y}) W^{\frac{1}{2}}(\tilde{x},\tilde{y},\tilde{z}) \quad (4.53)$$

Now, from (4.48a) and (4.48b),

$$\begin{aligned} u(\tilde{x},\tilde{y},\tilde{z}) &= u_{l_2}(\tilde{x},\tilde{y},\tilde{z}) - u_{l_1}(\tilde{y}) \\ &= -\hat{k}_d (C_r B)^{-1}\left[F_1 C[S_1\tilde{x} + B\tilde{y}] + C^*\tilde{z}\right] + \hat{k}_d \tilde{y} \\ &= -\hat{k}_d (C_r B)^{-1}\left[F_1 C[S_1\tilde{x} + B\tilde{y}] + C^*\tilde{z} - C_r B\tilde{y}\right] \\ &= -\hat{k}_d (C_r B)^{-1} C^* w(\tilde{x},\tilde{y},\tilde{z}) \end{aligned} \quad (4.54)$$

Thus, there exist calculable constants k_3, k_4 such that

$$\sup\ \{\langle \nabla_{\tilde{y}} W(\tilde{x},\tilde{y},\tilde{z}),\ h\rangle :\ h \in \overline{\mathbb{B}}_m(\lambda\|u(\tilde{x},\tilde{y},\tilde{z})\|)\} \le k_3 W(\tilde{x},\tilde{y},\tilde{z}) \quad (4.55)$$

and

$$\langle \nabla_{\tilde{y}} W(\tilde{x},\tilde{y},\tilde{z}),\ u(\tilde{x},\tilde{y},\tilde{z})\rangle \le k_4 W(\tilde{x},\tilde{y},\tilde{z}) \quad (4.56)$$

Combining (4.53), (4.55) and (4.56), the result follows.

Lemma 4.8

There exists a calculable constant η_1 such that for all $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^q$,

$$\sup \mathcal{G}_3(\tilde{x}, \tilde{y}, \tilde{z}) \leq \eta_1 V^{\frac{1}{2}}(\tilde{x}, \tilde{y}) W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z})$$

where

$$\mathcal{G}_3(\tilde{x}, \tilde{y}, \tilde{z}) := \{\langle \nabla_{\tilde{y}} V(\tilde{x}, \tilde{y}),\ u + h \rangle :\ h \in \overline{\mathbb{B}}_m(\lambda \| u(\tilde{x}, \tilde{y}, \tilde{z}) \|);$$

$$u = u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) - u_{l_1}(\tilde{y})\}$$

Proof (Sketch)

$\nabla_{\tilde{y}} V(\tilde{x}, \tilde{y}) = \tilde{y}$, and so $\|\nabla_{\tilde{y}} V(\tilde{x}, \tilde{y})\|$ is bounded above by a calculable scalar multiple of $V^{\frac{1}{2}}(\tilde{x}, \tilde{y})$. From Lemma 4.7 (i.e. equation (4.54)),

$$u(\tilde{x}, \tilde{y}, \tilde{z}) = u_{l_2}(\tilde{x}, \tilde{y}, \tilde{z}) - u_{l_1}(\tilde{y}) = -\hat{\kappa}_d (C_r B)^{-1} C^* w(\tilde{x}, \tilde{y}, \tilde{z}) \quad .$$

Thus, there exist calculable constants k_5, k_6 such that

$$\sup \{\langle \nabla_{\tilde{y}} V(\tilde{x}, \tilde{y}),\ h \rangle :\ h \in \overline{\mathbb{B}}_m(\lambda \| u(\tilde{x}, \tilde{y}, \tilde{z}) \|)\} \leq k_5 V^{\frac{1}{2}}(\tilde{x}, \tilde{y}) W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z})$$

and

$$\langle \nabla_{\tilde{y}} V(\tilde{x}, \tilde{y}),\ u(\tilde{x}, \tilde{y}, \tilde{z}) \rangle \leq k_6 V^{\frac{1}{2}}(\tilde{x}, \tilde{y}) W^{\frac{1}{2}}(\tilde{x}, \tilde{y}, \tilde{z}) \ ,$$

from which the result follows.

The next theorem establish that system (4.50) is globally uniformly asymptotically stable for all $\mu_d > 0$ sufficiently small.

Theorem 4.4

Let κ_d^* be defined as in Theorem 4.3 and let define

$$\mu_d^* := \frac{\alpha_1 \beta_0}{\alpha_1 \psi_3 + \eta_1(\theta_0 + \psi_4)} > 0 .$$

Then, for each fixed $\hat{\kappa}_d > \max\{\kappa_d^*, (1-\beta_2)^{-1}\gamma\}$ and fixed $\mu_d \in (0, \mu_d^*)$, the zero state of system (4.50) is globally uniformly asymptotically stable.

Proof

The multifunction $\mathbf{F}_2$ defined by (4.50b-d) is upper semi-continuous with convex and compact values. Hence, for each $(\tilde{x}(t_0), \tilde{y}(t_0), \tilde{z}(t_0)) \in \mathbb{R}^{n-m}\times\mathbb{R}^m\times\mathbb{R}^q$, there exists a local solution $(\tilde{x}, \tilde{y}, \tilde{z})\colon [t_0, \tau) \to \mathbb{R}^{n-m}\times\mathbb{R}^m\times\mathbb{R}^q$ to the system (4.50) (Aubin and Cellina 1984).

Now, define $\mathcal{W}_d\colon \mathbb{R}^{n-m}\times\mathbb{R}^m\times\mathbb{R}^q \to \mathbb{R}^+$, a Lyapunov function candidate, as

$$\mathcal{W}_d(\tilde{x}, \tilde{y}, \tilde{z}) := V(\tilde{x}, \tilde{y}) + \eta_1(\theta_0 + \psi_4)^{-1} W(\tilde{x}, \tilde{y}, \tilde{z}) ,$$

then, along every local solution $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}(\cdot))$ of (4.50), the following holds almost everywhere

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_d(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) = {} & \langle \nabla_{\tilde{x}} V(\tilde{x}(t), \tilde{y}(t)),\, f_1(\tilde{x}(t), \tilde{y}(t))\rangle \\ & + \sup \mathcal{G}_1(\tilde{x}(t), \tilde{y}(t)) + \sup \mathcal{G}_3(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \\ & + k_1 \big[\langle \nabla_{\tilde{x}} W(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)),\, f_1(\tilde{x}(t), \tilde{y}(t))\rangle \\ & + \sup \mathcal{G}_2(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))\end{aligned}$$

$$+ \mu_d^{-1}\langle \nabla_z W(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)),\ f_3(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))\rangle]$$

where $k_1 := \eta_1(\theta_0 + \psi_4)^{-1}$. By utilizing Lemmas 4.2,4.3,4.6-4.8, the following holds almost everywhere along every local solution $(\tilde{x}(\cdot),\tilde{y}(\cdot),\tilde{z}(\cdot))$ of (4.50),

$$\frac{d}{dt}W_d(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \le -\left\langle \begin{bmatrix} V^{\frac{1}{2}}(\tilde{x}(t),\tilde{y}(t)) \\ W^{\frac{1}{2}}(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \end{bmatrix}, M_{\mu_d} \begin{bmatrix} V^{\frac{1}{2}}(\tilde{x}(t),\tilde{y}(t)) \\ W^{\frac{1}{2}}(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \end{bmatrix} \right\rangle \tag{4.57a}$$

where

$$M_{\mu_d} := \begin{bmatrix} \alpha_1 & -\eta_1 \\ -\eta_1 & (\mu_d^{-1}\beta_0 - \psi_3)(\theta_0 + \psi_4)^{-1}\eta_1 \end{bmatrix}. \tag{4.57b}$$

Noting that M_{μ_d} is positive definite, then the theorem follows.

CHAPTER 5

ADAPTIVE STABILIZATION OF A CLASS OF UNCERTAIN SYSTEMS

5.1 Introduction

The proposed design approach given in the preceding chapter will work well if we have a suitable model that satisfies all the assumptions of the design. As we have seen in Chapter 4, the threshold values κ^* and μ^* (κ_d^* and μ_d^* in the discontinuous case) are crucial in this design and are explicitly calculable from known system data (i.e. in terms of known bounds of uncertainties). However, since these values are determined via a "worst case" analysis, it is to be expected that, in practice, the compensator will be conservative.

The main goal of this chapter is to develop adaptive-based feedback controls for a class of uncertain systems. This stabilizing adaptive version has a close relationship with compensator-based design proposed in the preceding chapter in the sense that the adaptive-based compensator is designed to circumvent the inherent conservatism induced by crude estimates in a "worst case" analysis. Furthermore, it can handle the case for which bounds on the uncertainties may be unknown (i.e. to allow for bounded uncertainties with unknown bounds). Thus, this adaptive-based design can be regarded as complementary to the compensator-based design.

In order to develop this adaptive compensator, we adopt a universal adaptive stabilization approach which is essentially that of Mårtensson (1985), but close in spirit to that of Ryan (1988); and akin to that of Corless and Leitmann

(1983, 1984).

This chapter is organized as follows. In § 5.2, we discuss state space representations for system (4.1,4.2) with filter dynamics. Section 5.3 deals with the adaptive stabilization by linear output feedback. Then, in § 5.4, the problem of stabilizing adaptive compensator by discontinuous output feedback will be considered, extending the adaptive compensator developed in preceding section. This is achieved (as in § 4.4) by admitting a discontinuous control component, modelled by a suitably chosen set-valued map, and overall controlled system is interpreted in the generalized sense of a controlled differential inclusions (Aubin and Cellina 1984). Finally, in § 5.5, we give example to illustrate the proposed approach.

5.2 State space representations

In order to proceed, we will give a state space representation for system (4.1,4.2) plus filter dynamics. Recall that the $(\tilde{x}, \tilde{y}, \tilde{z})$ representation used in §§ 4.3.2,4.4.3 (equation (4.23), and equation (4.50) in the discontinuous case) may be interpreted as follows.

For analysis only, we have separated the component $G_\mu(s)N(s)$ of the proposed compensator as two components $G_\mu(s)$ and $N(s)$, where the dynamic block $G_\mu(s) = \text{diag}\,\{\Psi_\mu^i(s)\}$ is realized by linear system $\Gamma^* = (C^*, \mu^{-1}A^*, \mu^{-1}B^*)$ with state dimension $\sum_{i=1}^{m}\delta_i$ where A^*, B^* and C^* are defined by (4.20). However, in practice, the component $G_\mu(s)N(s)$ is realized by constructing a total of mp filters of the form

$$\frac{n_{ij}(s)}{\chi_i(\mu s)}\ , \quad i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, p, \tag{5.1}$$

where $n_{ij}(s)$ denotes the ij-th element of N. Each filter of the form (5.1) can be interpreted as a single-input single-output system having a state space realization of the form

$$\mu\dot{\omega}(t) = A^i\omega(t) + B^i v(t) \tag{5.2a}$$

$$\gamma^i(t) = D_1^i(\mu)v(t) + D_2^i(\mu)\omega(t) \tag{5.2b}$$

Example 5.1

If

$$\frac{n_{ij}(s)}{\chi_i(\mu s)} = \frac{b_{\delta_i}s^{\delta_i} + b_{\delta_i-1}s^{\delta_i-1} + \cdots + b_1 s + b_0}{(\mu s)^{\delta_i} + a_{\delta_i-1}(\mu s)^{\delta_i-1} + \cdots + a_1(\mu s) + 1}$$

then

$$A^i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -a_2 & -a_3 & \cdots & -a_{\delta_i} \end{bmatrix} ; \; B^i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} ; \; D_1^i(\mu) = b_{\delta_i}\mu^{-\delta_i} \; ;$$

$$D_2^i(\mu) = [(b_0 - b_{\delta_i}\mu^{-\delta_i}) \; (b_1\mu^{-1} - a_2 b_{\delta_i}\mu^{-\delta_i}) \; \cdots$$

$$\cdots \; (b_{\delta_i-2}\mu^{-(\delta_i-2)} - a_{\delta_i-1}b_{\delta_i}\mu^{-\delta_i}) \; (b_{\delta_i-1}\mu^{-(\delta_i-1)} - a_{\delta_i}b_{\delta_i}\mu^{-\delta_i})] \; .$$

Thus, $G_\mu(s)N(s)$ has a state space realization in the form of a p-input, m-output linear system $\Gamma^\mu = (D_1(\mu), D_2(\mu), \mu^{-1}\mathcal{A}, \mu^{-1}\mathcal{B})$ with state dimension $\bar{q} = pq$ for which $\sigma(\mathcal{A}) \subset \mathbf{C}^-$ and the pair $(D_1(\mu), D_2(\mu))$ determines the output map, $D_1(\mu)$ being a feedforward operator. Therefore, the overall controlled system has the structure shown in Figure 5.1 below (Figure 5.2 is the structure of the associated discontinuous case).

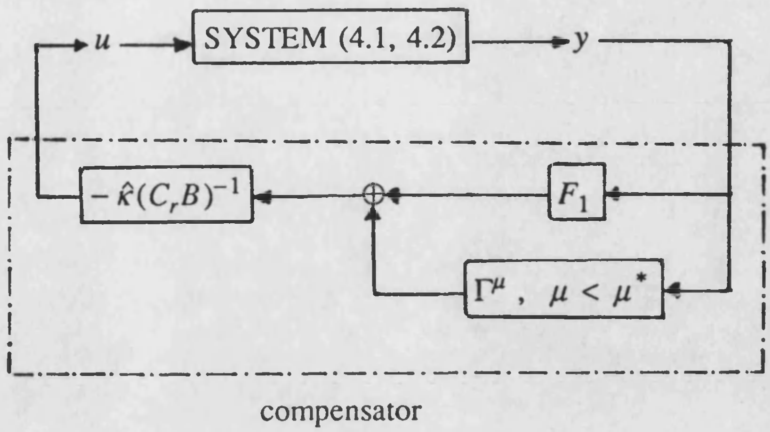

Figure 5.1. Linear case

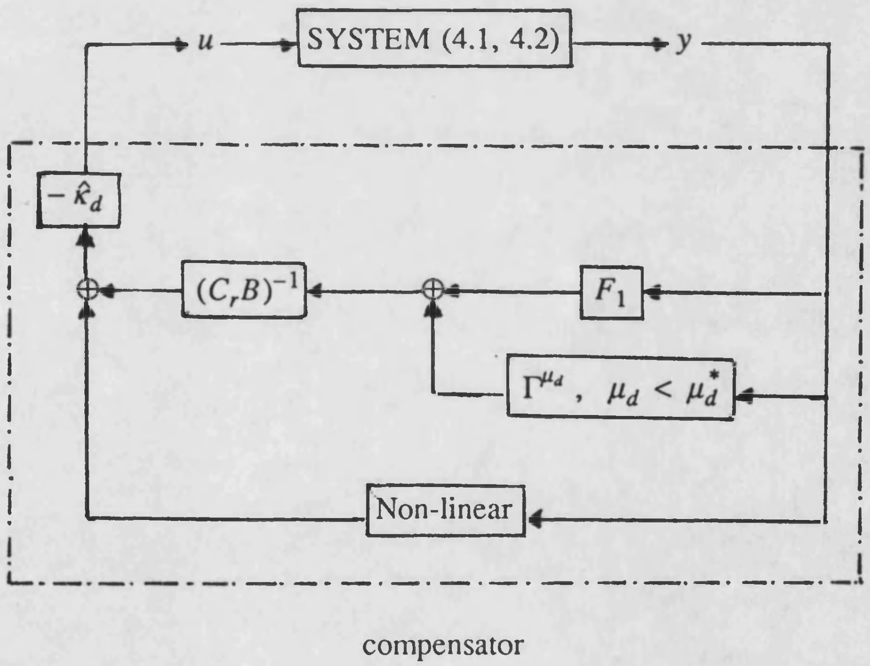

Figure 5.2. Discontinuous case

The governing equations (equivalent to (4.23)) can be expressed as

$$\dot{x}(t) = Ax(t) + B[u(t) + g(t,x(t),u(t))], \quad x(t) \in \mathbb{R}^n, \tag{5.3a}$$

$$\mu\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}y(t), \quad z(t) \in \mathbb{R}^{\bar{q}}, \quad \mu < \mu^*, \tag{5.3b}$$

$$y(t) = Cx(t), \quad y(t) \in \mathbb{R}^p, \tag{5.3c}$$

$$u(t) = -\hat{\kappa}(C_rB)^{-1}[F_1y(t) + D_1(\mu)y(t) + D_2(\mu)z(t)], \quad u(t) \in \mathbb{R}^m,$$

$$\hat{\kappa} > \kappa^*(1-\beta)^{-1}, \tag{5.3d}$$

and in the generalized feedback control case, the governing equations (equivalent to (4.50)) can be expressed as

$$\dot{x}(t) = Ax(t) + B[u(t) + g(t,x(t),u(t))], \quad x(t) \in \mathbb{R}^n, \tag{5.4a}$$

$$\mu_d\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}y(t), \quad z(t) \in \mathbb{R}^{\bar{q}}, \quad \mu_d < \mu_d^*, \tag{5.4b}$$

$$y(t) = Cx(t), \quad y(t) \in \mathbb{R}^p, \tag{5.4c}$$

$$u(t) \in -\hat{\kappa}_d[(C_rB)^{-1}[F_1y(t) + D_1(\mu)y(t) + D_2(\mu)z(t)] + \mathrm{N}(y(t))],$$

$$\hat{\kappa}_d > \max\{\kappa_d^*, (1-\beta_2)^{-1}\gamma\} \tag{5.4d}$$

Clearly, the threshold values κ^* and μ^* (κ_d^* and μ_d^* in the discontinuous case) are central to this design. Since these values are determined via a "worst case" analysis, it is to be expected that, in practical implementation, the compensator will be conservative. In the next section, a stabilizing adaptive version of the compensator is developed; however, in the case $r \geq 2$, this is achieved at the expense of imposing further structure on the uncertain function g.

Before proceeding, it is worth mentioning that this chapter should be read in conjunction with Chapter 4, since we are discussing a system with the basic assumptions (i.e. A4.1-A4.2); the only difference being in the structure of g.

5.3 Adaptive stabilization by linear output feedback

In this section, we will develop a stabilizing adaptive (linear) output feedback for system (5.3). This adaptive control requires only knowledge of $F_1, F_2, \cdots, F_r$ and C_rB. Thus, in the next sub-section, we first consider adaptive version for a special case ($r = 1$). Then, in § 5.3.2, a stabilizing adaptive compensator is developed by an approach which is essentially that of Mårtensson (1985).

The subject of discussion in this section can be found in Ryan and Yaacob (1989).

5.3.1 Case r = 1: Stabilizing adaptive output feedback for the true system (4.1,4.2)

If A4.2 holds with $r = 1$, then, by Corollary 4.1, system (4.1,4.2) is asymptotically stabilized by the static output feedback (4.8) with $\hat{\kappa} > \kappa^*(1-\beta)^{-1}$ provided, of course, that F_1 and C_rB are known and that sufficient *a priori* information is available to compute the (conservative) gain threshold $\kappa^*(1-\beta)^{-1}$. We now consider the case for which the latter information is unavailable, i.e. we only assume knowledge of F_1 and C_rB and, in particular, the constants α and $\beta < 1$ in A4.3 may be unknown. Assumptions A4.1 and A4.2 remain in force.

Replace fixed $\hat{\kappa}$ in (4.8) by variable $\kappa(t)$ to yield

$$u(t) = -\kappa(t)(C_rB)^{-1}F_1y(t) \tag{5.5a}$$

and let $\kappa(t)$ evolve according to the adaptation law

$$\dot{\kappa}(t) = \|(C_rB)^{-1}F_1y(t)\|^2 \tag{5.5b}$$

then,

Theorem 5.1

For all initial data $(t_0, x(t_0), \kappa(t_0)) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^+$, the adaptively controlled system (4.1,4.2,5.5) exhibits the following properties:

(i) $\lim_{t\to\infty} \kappa(t)$ exists and is finite;

(ii) $\lim_{t\to\infty} \|x(t)\| = 0$.

Proof

For fixed (but unknown) $\hat{\kappa} > \kappa^*(1-\beta)^{-1}$ and under the similarity transformation T, system (4.1,4.2,5.5) may be expressed as

$$\dot{\tilde{x}}(t) = A_{11}\tilde{x}(t) + A_{12}\tilde{y}(t) \tag{5.6a}$$

$$\dot{\tilde{y}}(t) = A_{21}\tilde{x}(t) + A_{22}\tilde{y}(t) - \hat{\kappa}\tilde{y}(t) - [\kappa(t) - \hat{\kappa}]\tilde{y}(t)$$

$$+ \tilde{g}(t, \tilde{x}(t), \tilde{y}(t), -\kappa(t)\tilde{y}(t)) \tag{5.6b}$$

$$\dot{\kappa}(t) = \|\tilde{y}(t)\|^2 \tag{5.6c}$$

with $(\tilde{x}(t_0), \tilde{y}(t_0), \kappa(t_0)) = (\tilde{x}_0, \tilde{y}_0, \kappa_0)$.

Let U and V be as in the proof of Theorem 4.1 and define the positive definite (since $\beta < 1$) function

$$\mathcal{V}\colon (\tilde{x}, \tilde{y}, \kappa) \mapsto V(\tilde{x}, \tilde{y}) + \tfrac{1}{2}(\kappa - \hat{\kappa})^2 - \tfrac{1}{2}\beta(\kappa - \hat{\kappa})\,|\kappa - \hat{\kappa}|\ . \tag{5.7}$$

Then, along solutions $(\tilde{x}(\cdot), \tilde{y}(\cdot), \kappa(\cdot))$ of (5.6), the following holds almost everywhere

$$\frac{d}{dt}\mathcal{V}(\tilde{x}(t),\tilde{y}(t),\kappa(t)) \le -U(\tilde{x}(t),\tilde{y}(t)) - \beta\hat{\kappa}\|\tilde{y}(t)\|^2$$

$$- [\kappa(t)-\hat{\kappa}]\|\tilde{y}(t)\|^2 + \beta\kappa(t)\|\tilde{y}(t)\|^2$$

$$+ \big[[\kappa(t)-\hat{\kappa}] - \beta|\kappa(t)-\hat{\kappa}|\big]\|\tilde{y}(t)\|^2$$

$$\le -U(\tilde{x}(t),\tilde{y}(t)) \tag{5.8}$$

Since U is positive definite, we conclude that $t \mapsto (\tilde{x}(t),\tilde{y}(t),\kappa(t))$ is bounded and since $t \mapsto \kappa(t)$ is also monotonic, assertion (i) of the theorem follows.

Furthermore, in view of (5.8), for solutions $(\tilde{x},\tilde{y},\kappa)\colon [t_0,\infty) \to \mathbb{R}^{n-m}\times\mathbb{R}^m\times\mathbb{R}$ of (5.6),

$$\int_{t_0}^{\infty} U(\tilde{x}(t),\tilde{y}(t))\,dt \le \mathcal{V}(\tilde{x}_0,\tilde{y}_0,\kappa_0) < \infty \tag{5.9}$$

Hence, since U and V are positive definite forms,

$$\int_{t_0}^{\infty} V(\tilde{x}(t),\tilde{y}(t))\,dt < \infty \tag{5.10}$$

Furthermore, (5.8) ensures that there exists a constant $c(\tilde{x}_0,\tilde{y}_0) > 0$ such that

$$\dot{V}(\tilde{x}(t),\tilde{y}(t)) \le c(\tilde{x}_0,\tilde{y}_0) \tag{5.11}$$

Invoking Lemma 6.3 of Corless and Leitmann (1984), we conclude (from (5.10) and (5.11)) that $V(\tilde{x}(t),\tilde{y}(t)) \to 0$ as $t \to \infty$ whence assertion (ii) of the theorem.

5.3.2 Cases r ≥ 2: Stabilizing adaptive compensator for system (5.3)

Before describing the adaptive strategy in this case, it is remarked that the argument used in establishing Theorem 5.1 cannot be carried over directly. Instead, we will base our approach on that of Mårtensson (1985). For this reason, further conditions are imposed on the uncertain function g, i.e. g depends linearly on x. In particular, A4.3 is now replaced by:

A5.1: There exist a bounded continuous function $\Delta A\colon \mathbb{R} \to \mathbb{R}^{m \times n}$, a Carathéodory function $g_3\colon \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$, and a constant β such that for all $(t, x, u) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$,

(i) $g(t, x, u) = \Delta A(t)x + g_3(t, u)$;

(ii) $\|g_3(t, u)\| \leq \beta\|u\|, \ \beta < 1$;

(iii) $(C, A + B\Delta A(\cdot))$ is uniformly completely observable in the sense of Definition 2.8.

Note that, if A5.1 holds, then A4.3 holds *a fortiori* with $\alpha = \sup_t \|\Delta A(t)\|$ provided that α, and β are known. However, knowledge of these constants is *not* required here.

Example 5.2

With (C, A, B) defined as in Example 4.1 of Chapter 4, A5.1 holds for any bounded continuous $\Delta A\colon t \mapsto (\Delta a_1(t), \Delta a_2(t), \Delta a_3(t))$.

Now replace fixed $\hat{\kappa}$ in (5.3d) by variable $\kappa(t) > 0$ and replace fixed μ in (5.3b) by $(\delta\kappa(t))^{-1}$, where $\delta > 0$ is a constant (design parameter) and let $\kappa(t)$ evolve according to the adaptation law (other adaptation laws may be feasible, as discussed in Ilchmann *et al.* 1987)

$$\dot{\kappa}(t) = \|y(t)\|^2 + \|z(t)\|^2 \tag{5.12}$$

Writing (as in Mårtensson 1985)

$$x_a(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad u_a(t) = \begin{bmatrix} u(t) \\ \dot{z}(t) \end{bmatrix}, \quad y_a(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \tag{5.13}$$

then the overall adaptively controlled system may be expressed in the form

$$\dot{x}_a(t) = A_a(t)x_a(t) + B_a[u_a(t) + g_a(t, u_a(t))], \quad x_a(t) \in \mathbb{R}^{n+\bar{q}}, \tag{5.14a}$$

$$y_a(t) = C_a x_a(t), \quad y_a(t) \in \mathbb{R}^{p+\bar{q}}, \tag{5.14b}$$

$$u_a(t) = -\kappa(t)K_a(\kappa(t))y_a(t), \quad u_a(t) \in \mathbb{R}^{m+\bar{q}}, \tag{5.14c}$$

$$\dot{\kappa}(t) = \|y_a(t)\|^2, \tag{5.14d}$$

where

$$A_a(t) := \begin{bmatrix} A + B\Delta A(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad B_a := \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad C_a := \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \tag{5.14e}$$

$$K_a(\kappa) := \begin{bmatrix} (C_rB)^{-1}[F_1 + D_1((\delta\kappa)^{-1})] & (C_rB)^{-1}D_2((\delta\kappa)^{-1}) \\ -\delta\mathcal{B} & -\delta\mathcal{A} \end{bmatrix}, \tag{5.14f}$$

$$g_a(t, u_a) := \begin{bmatrix} g_3(t, u) \\ 0 \end{bmatrix}. \tag{5.14g}$$

The stability of system (5.14) will now be investigated. We first require the following lemma (essentially a non-autonomous version of Mårtensson's lemma (Mårtensson 1985)).

Lemma 5.1

Let x_a: $\mathbb{R} \to \mathbb{R}^{n+\bar{q}}$ satisfy

$$\dot{x}_a(t) = A_a(t)x_a(t) + B_a[v(t) + g_a(t, v(t))] \tag{5.15}$$

where v: $\mathbb{R} \to \mathbb{R}^{m+\bar{q}}$ is measurable. Then, there exist constants c_a, $\tau > 0$ such that for all t,

$$\|x_a(t)\|^2 \le c_a \int_{t-\tau}^{t} [\|y_a(s)\|^2 + \|v(s)\|^2]\, ds\, . \tag{5.16}$$

Proof

Let $\Phi(\cdot,\cdot)$ denote the state transition matrix function generated by $A + B\Delta A(\cdot)$ and define the observability Gramian for the pair $(C, A + B\Delta A(\cdot))$ in the usual manner, i.e.

$$\mathrm{M}(t,s) := \int_{s}^{t} \Phi^T(\sigma, s)C^T C\Phi(\sigma, s)\, d\sigma\, . \tag{5.17}$$

Now, for some constants λ_1 and ω, we have $\|\exp At\| \le \lambda_1 \exp(\omega t)$ and, since $\Delta A(\cdot)$ is bounded (by assumption), there exists a constant λ_2 such that $\|B\Delta A(t)\| \le \lambda_2$. By standard perturbation theory, we conclude that,

$$\|\Phi(t,s)\| \le \lambda_1 \exp\,[(\omega + \lambda_1\lambda_2)(t-s)], \ \text{ for all } t, s\, . \tag{5.18}$$

Clearly, the state transition matrix function $\Phi_a(\cdot,\cdot)$ generated by $A_a(\cdot)$ is given by

$$\Phi_a(t,s) = \begin{bmatrix} \Phi(t,s) & 0 \\ 0 & I \end{bmatrix}, \tag{5.19}$$

whence

$$\|\Phi_a(t,s)\| \le \psi(t-s)\, , \ \text{ for all } t, s\, , \tag{5.20a}$$

where

$$\psi\colon \sigma \mapsto 1 + \lambda_1 \exp\,[(\omega + \lambda_1\lambda_2)\sigma] \tag{5.20b}$$

The observability Gramian for the pair $(C_a, A_a(\cdot))$ is given by

$$\mathrm{M}_a(t,s) := \int_s^t \Phi_a^T(\sigma,s) C_a^T C_a \Phi_a(\sigma,s)\, ds = \begin{bmatrix} \mathrm{M}(t,s) & 0 \\ 0 & (t-s)I \end{bmatrix}, \quad (5.21)$$

and, since $(C, A+B\Delta A(\cdot))$ is uniformly completely observable (by assumption), we may conclude (see Definition 2.8) that there exist positive constants $\tau,\ c_1,\ c_2$ such that, for all t,

$$c_1 \|\zeta\|^2 \leq \langle \zeta,\ \mathrm{M}_a(t,t-\tau)\,\zeta\rangle \leq c_2 \|\zeta\|^2, \quad \text{for all } \zeta \in \mathbb{R}^{n+\bar{q}}. \quad (5.22)$$

Now define the measurable function $v_a \colon t \mapsto v(t) + g_a(t, v(t))$ and note that $\|v_a(t)\| \leq (1+\beta)\|v(t)\|$. Then,

$$x_a(t) = \Phi_a(t,t-\tau)x_a(t-\tau) + \int_{t-\tau}^t \Phi_a(t,s)B_a v_a(s)\, ds \quad (5.23)$$

whence

$$\|x_a(t)\|^2 \leq 2\|\Phi_a(t,t-\tau)x_a(t-\tau)\|^2 + 2\|\int_{t-\tau}^t \Phi_a(t,s)B_a v_a(s)\, ds\|^2$$

$$\leq 2c_3\|x_a(t-\tau)\|^2 + 2c_4(1+\beta)^2\|B_a\|^2 \int_{t-\tau}^t \|v(s)\|^2\, ds, \quad (5.24a)$$

wherein (5.20) has been used, and

$$c_3 := \psi^2(\tau), \quad c_4 := \int_0^\tau \psi^2(s)\, ds. \quad (5.24b)$$

Also, invoking (5.14b), (5.20), (5.22) and (5.23), we have

$$\|x_a(t-\tau)\|^2 \leq c_1^{-1}\langle x_a(t-\tau), \mathrm{M}_a(t,t-\tau)x_a(t-\tau)\rangle$$

$$= c_1^{-1}\int_{t-\tau}^t \|C_a\Phi_a(s,t-\tau)x_a(t-\tau)\|^2\, ds$$

$$= c_1^{-1}\int_{t-\tau}^t \|y_a(s) - C_a\int_{t-\tau}^s \Phi_a(s,\sigma)B_a v_a(\sigma)\, d\sigma\|^2\, ds$$

$$\leq 2c_1^{-1}\Big[\int_{t-\tau}^t \|y_a(s)\|^2\, ds$$

$$+ c_5\tau(1+\beta)^2\|C_a\|^2\|B_a\|^2\int_{t-\tau}^t \|v(s)\|^2\, ds\Big], \quad (5.25a)$$

where

$$c_5 := \int_0^{\tau}\int_0^{s} \psi^2(\sigma)\, d\sigma\, ds\,. \tag{5.25b}$$

Combining (5.24) and (5.25) yields the required result.

Now we state and prove the stability theorem for the system (5.14).

Theorem 5.2

For all initial data $(t_0, x_a(t_0), \kappa(t_0)) \in \mathbb{R}\times\mathbb{R}^{n+\bar{q}}\times(0,\infty)$, system (5.14) exhibits the following properties:

(i) $\lim_{t\to\infty} \kappa(t)$ exists and is finite;

(ii) $\lim_{t\to\infty} \|x_a(t)\| = 0$.

Proof

Seeking a contradiction to (i), suppose that the monotonically increasing function $t \mapsto \kappa(t)$ is unbounded. Then, for some $t_1 \in [0,\infty)$, $\kappa(t_0+t_1) = \hat{\kappa} > \kappa^*(1-\beta)^{-1}$ and $(\delta\kappa(t_0+t_1))^{-1} = \mu < \mu^*$. Now, an argument similar to that used in the proof of Theorem 4.2 can be adopted to establish that $x(\cdot)$ (and hence $y(\cdot) = Cx(\cdot)$) must ultimately tend exponentially to zero (and hence are square integrable on $[t_0,\infty)$).

Consider now the filter equation part of (5.14c), i.e.

$$\dot{z}(t) = \delta\kappa(t)[\mathcal{A}z(t) + \mathcal{B}y(t)] \tag{5.26}$$

Let φ_1 (with inverse φ_1^{-1}) denote the monotonic function $t \mapsto \int_{t_0}^{t} \delta\kappa(s)\, ds$. Then, it can be shown that

$$z(t) = \exp(\mathcal{A}\varphi_1(t))z(t_0) + \int_0^{\varphi_1(t)} \exp[\mathcal{A}(\varphi_1(t)-s)]\mathcal{B}y(\varphi_1^{-1}(s))\, ds \tag{5.27}$$

satisfies (5.26). Since $y(\cdot)$ is exponentially tend to zero, $y(\varphi_1^{-1}(\cdot))$ is clearly bounded. Since $\sigma(\mathcal{A}) \subset \mathbb{C}^-$, we may conclude from (5.27) that z is bounded. Hence, from (5.14d), $\dot{\kappa}(t)$ is bounded and so there exists a constant κ_1 such that

$$\kappa(t) \le \kappa(t_0) + \kappa_1(t-t_0), \quad \text{for all } t \ge t_0 \,. \tag{5.28}$$

Now, it is readily verified that the function $y(\varphi_1^{-1}(\cdot))$ ultimately satisfies

$$\|y(\varphi_1^{-1}(s))\| \le \kappa_2 \exp\left[\kappa_3 - \sqrt{(\kappa_3^2 + \kappa_4 s)}\right] \tag{5.29}$$

for some positive constants κ_i ($i = 2, 3, 4$), and so is square integrable on $[t_0, \infty)$. From (5.27) (since $\sigma(\mathcal{A}) \subset \mathbb{C}^-$) we may conclude that $z(\cdot)$ is square integrable on $[t_0, \infty)$. Thus, $y_a(\cdot)$ is square integrable on $[t_0, \infty)$ which, in view of (5.14d), contradicts our supposition that the function κ is unbounded. This establishes assertion (i) of the theorem.

It remains to show that $x_a(t) \to 0$ as $t \to \infty$. Clearly, (i) ensures that y_a is square integrable on $[t_0, \infty)$ and, in view of (5.14c), that u_a is a bounded linear transformation of y_a. Thus, we may conclude that u_a is also square integrable on $[t_0, \infty)$. Now, by Lemma 5.1, we have

$$\begin{aligned}
\|x_a(t)\|^2 &\le c_a \int_{t-\tau}^{t} [\|y_a(s)\|^2 + \|u_a(s)\|^2]\, ds \\
&= c_a \int_{t_0}^{t} [\|y_a(s)\|^2 + \|u_a(s)\|^2]\, ds \\
&\quad - c_a \int_{t_0}^{t-\tau} [\|y_a(s)\|^2 + \|u_a(s)\|^2]\, ds
\end{aligned} \tag{5.30}$$

Therefore, $\|x_a(t)\| \to 0$ as $t \to \infty$.

5.4 Adaptive stabilization by discontinuous output feedback

This section considers the problem of adaptive stabilization of a class of uncertain systems by discontinuous output feedback. Our goal is to extend the adaptive strategy developed in § 5.3, by expanding the class of allowable uncertainties. A generalized adaptive output feedback strategy is developed which renders the zero state globally attractive. The generalized strategy has a linear plus discontinuous output feedback structure with bounded adaptive scalar gain. An appropriately chosen set-valued map models the discontinuous control component and we adopt the analytic framework of controlled differential inclusions (Aubin and Cellina 1984).

In essence, the approach adopted here also is that of Mårtensson (1985) and in a similar ideas with that of Ryan (1988). Thus, here we attempt to expand Ryan (1988) to the cases $r \geq 2$, by using Mårtensson's method. However, this generalization is achieved at the expense of extra assumptions on the uncertain function g; and this will be discussed in § 5.4.2.

5.4.1 Case r = 1: Stabilizing generalized adaptive output feedback for the true system (4.1,4.2)

Recalling from § 4.4.2 that, if A4.2 holds with $r = 1$, then, by Corollary 4.2, for each fixed $\hat{k}_d > \max\{\kappa_d^*, (1-\beta_2)^{-1}\gamma\}$ the generalized static output feedback (4.43) asymptotically stabilizes system (4.1,4.2) provided that, F_1 and C_rB are known and that sufficient *a priori* information is available to calculate the (conservative) gain threshold: $\max\{\kappa_d^*, (1-\beta_2)^{-1}\gamma\}$. We now consider the case for which the latter information is unavailable, i.e. we only assume knowledge of F_1 and C_rB; in particular, the constants $\alpha, \beta_1 < 1, \beta_2 < 1$ and γ

in A4.4 may be unknown. Assumptions A4.1 and A4.2 remain in force.

Replace fixed $\hat{\kappa}_d$ in (4.43) by variable $\kappa_d(t)$ to yield the generalized feedback

$$u(t) \in -\kappa_d(t)\left[(C_rB)^{-1}F_1y(t) + \mathrm{N}(y(t))\right] \tag{5.31a}$$

where the set-valued map $y \mapsto \mathrm{N}(y) \subset \mathbb{R}^m$ is defined as in (4.37b), and $\kappa_d(t)$ evolves according to the adaptation law

$$\dot{\kappa}_d(t) = \left[\|(C_rB)^{-1}F_1y(t)\| + \xi(y(t))\right]\|(C_rB)^{-1}F_1y(t)\| \tag{5.31b}$$

then, for completeness, we state (without proof) the following lemma (see Ryan 1988, Theorem 2)

Lemma 5.2

For all initial data $(t_0, x(t_0), \kappa_d(t_0)) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^+$, the adaptive output feedback system (4.1,4.2,5.31) possesses the following properties:

(i) existence and continuation of solutions;

(ii) $\lim_{t\to\infty} \kappa_d(t)$ exists and is finite;

(iii) $\lim_{t\to\infty} \|x(t)\| = 0$.

5.4.2 Cases r ≥ 2: Stabilizing generalized adaptive compensator for system (5.4)

In this sub-section, we consider the case for which *a priori* information is unavailable to calculate the (conservative) gain threshold $\max\{\kappa_d^*, (1-\beta_2)^{-1}\gamma\}$ in Theorem 4.3 of the preceding chapter, i.e. we only assume knowledge of F_i, $i = 1, 2, \cdots, r$, and C_rB, and the constants

α, β_1, β_2 and γ in A4.4 may be unknown. We adopt the approach of Mårtensson (1985) and for this reason, we have to impose further conditions on the uncertain function g. Here, we need that "$(I-\Pi)$" part of g_1 is assumed to depend linearly on x and g_2 is assumed to depend linearly on u. To be precise, A4.4 is now replaced by A5.2 below. All other assumptions (i.e. A4.1-A4.2) remain in force.

A5.2: There exist a non-negative constant γ, a bounded measurable function $\Delta A_1 : \mathbb{R} \to \mathbb{R}^{m \times n}$, a Carathéodory function $g_1 : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$, a known continuous function $\xi : \mathbb{R}^p \to \mathbb{R}^+$, and matrices $\Delta B, \Delta B_1, \Delta B_2 \in \mathbb{R}^{m \times m}$ such that, for all (t, x, u),

(i) $g(t,x,u) = (I-\Pi)\Delta A_1(t)x + \Pi g_1(t,x) + \Delta B u$;

(ii) $\|\Pi g_1(t,x)\| \le \gamma\xi(Cx)$;

(iii) $\Delta B = (I-\Pi)\Delta B_1(I-\Pi) + \Pi\Delta B_2\Pi$, $\|\Delta B_1\| < 1$, $\|\Delta B_2\| < 1$;

(iv) $(C, A+B(I-\Pi)\Delta A_1(\cdot))$ is uniformly completely observable in the sense of Definition 2.8;

furthermore, if we define the class of exponentially bounded continuous functions Ξ by

$$\Xi := \{\eta : \mathbb{R} \to \mathbb{R}^p \mid \|\eta(t)\| \le M_0 e^{-\omega_0 t} \text{ for all } t \text{ and some } M_0 > 0, \omega_0 > 0\}$$

then,

(v) for each $\eta \in \Xi$, the composite function $\xi\circ\eta$ is square integrable on $[t_0, \infty)$, for all $t_0 \in \mathbb{R}$.

Remark

If A5.2 holds, then A4.4 holds with $\alpha = \sup_t \|\Delta A_1(t)\|$, $\beta_1 = \|\Delta B_1\| < 1$ and $\beta_2 = \|\Delta B_2\| < 1$ (since from A5.2 (iii) and using decomposition $u = (I - \Pi)u + \Pi u$, we have $\Delta Bu = (I - \Pi)\Delta B_1(I - \Pi)u + \Pi\Delta B_2\Pi u$), provided that α, β_1, β_2 and γ are known. However, knowledge of this constants is not required here.

Now replace fixed $\hat{\kappa}_d$ in (5.4d) by variable $\kappa_d(t) > 0$ and replace fixed μ_d in (5.4b) by $(\varepsilon\kappa_d(t))^{-1}$, where $\varepsilon > 0$ is a constant (design parameter) and let $\kappa_d(\cdot)$ generated via the adaptation law

$$\dot{\kappa}_d(t) = \|y(t)\|^2 + \|z(t)\|^2 + \xi^2(y(t)), \tag{5.32}$$

and writing (as in Mårtensson 1985)

$$x_d(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad u_d(t) = \begin{bmatrix} u(t) \\ \dot{z}(t) \end{bmatrix}, \quad y_d(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \tag{5.33}$$

then the overall adaptively controlled system may be written in the form

$$\dot{x}_d(t) = A_d(t)x_d(t) + B_d[(I + \Delta B_d)u_d(t) + g_d(t, x_d(t))], \quad x_d(t) \in \mathbb{R}^{n+\bar{q}}, \tag{5.34a}$$

$$y_d(t) = C_d x_d(t), \quad y_d(t) \in \mathbb{R}^{p+\bar{q}}, \tag{5.34b}$$

$$u_d(t) \in -\kappa_d(t)\left[K_d(\kappa_d(t))y_d(t) + \mathrm{N}_d(y_d(t))\right], \tag{5.34c}$$

$$\dot{\kappa}_d(t) = \|y_d(t)\|^2 + \xi^2(y(t)), \tag{5.34d}$$

where

$$A_d(t) := \begin{bmatrix} A+B(I-\Pi)\Delta A_1(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad B_d := \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \tag{5.34e}$$

$$\Delta B_d := \begin{bmatrix} \Delta B & 0 \\ 0 & 0 \end{bmatrix}, \quad C_d := \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \tag{5.34f}$$

$$K_d(\kappa_d) := \begin{bmatrix} (C_rB)^{-1}[F_1 + D_1((\varepsilon\kappa_d)^{-1})] & (C_rB)^{-1}D_2((\varepsilon\kappa_d)^{-1}) \\ -\varepsilon\mathcal{B} & -\varepsilon\mathcal{A} \end{bmatrix}, \tag{5.34g}$$

$$g_d(t,x_d) := \begin{bmatrix} \Pi g_1(t,x) \\ 0 \end{bmatrix}, \quad \mathrm{N}_d(y_d) := \mathrm{N}_d([C \;\; 0]x_d) = \begin{bmatrix} \mathrm{N}(y) \\ 0 \end{bmatrix} \tag{5.34h}$$

We are now going to investigate the stability of system (5.34). Since we wish to admit discontinuous feedback (as in the § 4.4), we need to recast the problem in the context of controlled differential inclusion system as follows.

Let define multifunction H_3 by

$$\mathrm{H}_3(x_d, \kappa_d) := -\kappa_d \left[K_d C_d x_d + \mathrm{N}_d([C \;\; 0]x_d)\right] \tag{5.35}$$

and let define multifunctions $\mathbf{D}_d$ and $\mathbf{F}_d$ as

$$\mathbf{D}_d(t,x_d,\kappa_d) := \{A_d(t)x_d + B_d[(I+\Delta B_d)v + g_d(t,x_d)] : v \in \mathrm{H}_3(x_d,\kappa_d)\} \tag{5.36}$$

$$\mathbf{F}_d(t,x_d,\kappa_d) := \mathbf{D}_d(t,x_d,\kappa_d)\times\{\|C_d x_d\|^2 + \xi^2([C \;\; 0]x_d)\} \tag{5.37}$$

Then, the controlled system (5.34) may be replaced by a controlled differential inclusion system

$$(\dot{x}_d(t), \dot{\kappa}_d(t)) \in \mathbf{F}_d(t, x_d(t), \kappa_d(t)) \tag{5.38}$$

Certainly, any generalized solution of (5.34) (satisfying (5.34) a.e.) is also a generalized solution of (5.38) (satisfying (5.38) a.e.).

Now, let $(x_d(\cdot), \kappa_d(\cdot))$ be a solution of (5.38). We first want to show that there exists $u^*(\cdot)$ such that $(x_d(\cdot), \kappa_d(\cdot), u^*(\cdot))$ is also a solution of (5.34). We show this by an argument similar that used in Dorling and Ryan (1985), and is reiterated in Lemma 5.3 below. For this purpose, by writing $B_\Delta = B_d(I + \Delta B_d)$ the systems (5.38) and (5.34) may be rewritten respectively as

$$\dot{x}_d(t) - A_d(t)x_d(t) - B_d g_d(t, x_d(t)) \in B_\Delta \mathrm{H}_3(x_d(t), \kappa_d(t)), \qquad (5.39a)$$

$$\dot{\kappa}_d(t) = \|C_d x_d(t)\|^2 + \xi^2([C \quad 0]x_d(t)), \qquad (5.39b)$$

and

$$\dot{x}_d(t) - A_d(t)x_d(t) - B_d g_d(t, x_d(t)) = B_\Delta u_d(t), \qquad (5.40a)$$

$$\dot{\kappa}_d(t) = \|C_d x_d(t)\|^2 + \xi^2([C \quad 0]x_d(t)), \qquad (5.40b)$$

$$u_d(t) \in \mathrm{H}_3(x_d(t), \kappa_d(t)), \qquad (5.40c)$$

Then, we may state the following lemma.

Lemma 5.3

Let $(x_d(\cdot), \kappa_d(\cdot))$ solve system (5.39). Then there exists a measurable function $u_d(\cdot) = u^*(\cdot)$ such that $(x_d(\cdot), \kappa_d(\cdot), u^*(\cdot))$ solves system (5.40).

Proof

Let $(x^*(\cdot), \kappa^*(\cdot))$ satisfy (5.39). Then define $t \mapsto u^*(t)$ by

$$u^*(t) = (B_\Delta^T B_\Delta)^{-1} B_\Delta^T [\dot{x}^*(t) - A_d(t)x^*(t) - B_d g_d(t, x^*(t))] \quad \text{a.e.} \qquad (5.41)$$

Note that $u^*(\cdot)$ defined above is well defined, by recalling that $x^*(\cdot)$ is absolutely continuous and hence differentiable almost everywhere, and $A_d(\cdot)$ and $g_d(\cdot, x^*(\cdot))$ are measurable and B_Δ has full rank m for almost all t. Then, we conclude that u^* is a measurable selection for $\mathrm{H}_3(x_d(\cdot), \kappa_d(\cdot))$. This, can be easily seen, since from (5.39),

$$\dot{x}^*(t) - A_d(t)x^*(t) - B_d g_d(t, x^*(t)) = B_\Delta h(t) \quad \text{a.e.} \qquad (5.42)$$

for some $h(t) \in \mathrm{H}_3(x_d(t), \kappa_d(t))$, and hence

$$\begin{aligned} u^*(t) &= (B_\Delta^T B_\Delta)^{-1} B_\Delta^T [\dot{x}^*(t) - A_d(t)x^*(t) - B_d g_d(t, x^*(t))] \\ &= (B_\Delta^T B_\Delta)^{-1} B_\Delta^T B_\Delta h(t) \\ &= h(t) \in \mathrm{H}_3(x_d(t), \kappa_d(t)) \quad \text{a.e.} \end{aligned} \qquad (5.43)$$

Now, by putting $x_d = x^*$ in (5.40) and use the fact that $B_\Delta(B_\Delta^T B_\Delta)^{-1} B_\Delta^T$ projects orthogonally onto im B_Δ, we have

$$\begin{aligned} \dot{x}^*(t) - A_d x^*(t) - B_d g_d(t, x^*(t)) &= B_\Delta(B_\Delta^T B_\Delta)^{-1} B_\Delta^T \\ &\quad .[\dot{x}^*(t) - A_d(t)x^*(t) - B_d g_d(t, x^*(t))] \\ &= B_\Delta u^*(t) \quad \text{a.e.}, \end{aligned} \qquad (5.44)$$

that is $(x^*(\cdot), \kappa^*(\cdot))$ solves (5.40) with $u_d = u^*$, which completes the proof.

Remark

As a consequence of Lemma 5.3, we may conclude that

$$\|u^*(t)\|^2 \le 2\kappa_d^2(t) \, [\|K_d(\kappa_d(t))\|^2 \|y_d(t)\|^2 + \xi^2(y(t))] \qquad (5.45)$$

Before stating and proving the main theorem of this section (i.e. the stability theorem for system (5.38)), we need the following lemma (essentially a generalized non-autonomous version of Mårtensson's lemma (Mårtensson 1985), and hence a generalized version of Lemma 5.1).

Lemma 5.4

Let (x_d, κ_d): $[t_0, \omega_1) \rightarrow \mathbb{R}^{n+\bar{q}} \times \mathbb{R}$ solve differential inclusion system (5.38) and let $u^*(\cdot)$ be defined as in (5.41). Then, there exist constants c_d, $\tau > 0$ such that, for all $t \in (t_0 + \tau, \omega_1)$,

$$\|x_d(t)\|^2 \leq c_d \int_{t-\tau}^{t} [\|y_d(s)\|^2 + \|u^*(s)\|^2 + \xi^2(y(s))]\, ds\,.$$

Proof (This lemma is proved in a similar manner that we prove Lemma 5.1)

Let $\Theta(\cdot,\cdot)$ be the state transition matrix function generated by $A + B(I - \Pi)\Delta A_1(\cdot)$ and define the observability Gramian for the pair $(C, A + B(I - \Pi)\Delta A_1(\cdot))$ by

$$\Lambda(t,s) := \int_s^t \Theta^T(\sigma, s) C^T C \Theta(\sigma, s)\, d\sigma\,. \tag{5.46}$$

Now, for some constants λ_1 and ω, we have $\|\exp At\| \leq \lambda_1 \exp(\omega t)$ and, since $\Delta A_1(\cdot)$ is bounded (by assumption), there exists a constant λ_3 such that $\|B(I - \Pi)\Delta A_1(t)\| \leq \lambda_3$. By standard perturbation theory, it can be shown that

$$\|\Theta(t,s)\| \leq \lambda_1 \exp\left[(\omega + \lambda_1\lambda_3)(t-s)\right], \quad \text{for all } t,\, s\,. \tag{5.47}$$

Clearly, the state transition matrix function $\Theta_d(\cdot,\cdot)$ generated by $A_d(\cdot)$ is given by

$$\Theta_d(t,s) = \begin{bmatrix} \Theta(t,s) & 0 \\ 0 & I \end{bmatrix}, \tag{5.48}$$

and hence

$$\|\Theta_d(t,s)\| \leq \psi_d(t-s), \quad \text{for all } t,\, s\,, \tag{5.49a}$$

where

$$\psi_d : \sigma \mapsto 1 + \lambda_1 \exp\left[(\omega + \lambda_1\lambda_3)\sigma\right] \tag{5.49b}$$

The observability Gramian for the pair $(C_d, A_d(\cdot))$ is given by

$$\Lambda_d(t,s) := \int_s^t \Theta_d^T(\sigma,s) C_d^T C_d \Theta_d(\sigma,s)\, d\sigma = \begin{bmatrix} \Lambda(t,s) & 0 \\ 0 & (t-s)I \end{bmatrix}, \quad (5.50)$$

and, since $(C, A + B(I-\Pi)\Delta A_1(\cdot))$ is uniformly completely observable (by assumption), we may conclude from Definition 2.8 that, there exist positive constants τ, c_6, c_7 such that for all $t \in (t_0+\tau, t_1)$,

$$c_6 \|\zeta\|^2 \le \langle \zeta, \Lambda_d(t, t-\tau)\zeta \rangle \le c_7 \|\zeta\|^2, \quad \text{for all } \zeta \in \mathbb{R}^{n+\bar{q}}. \quad (5.51)$$

Now, let $u^*(\cdot)$ be defined as in (5.41). Then,

$$x_d(t) = \Theta_d(t, t-\tau) x_d(t-\tau)$$

$$+ \int_{t-\tau}^t \Theta_d(t,s) B_d [(I + \Delta B_d) u^*(s) + g_d(s, x_d(s))]\, ds \quad (5.52)$$

Thus, using (5.49) and the fact that $\|I + \Delta B_d\| \le (1+\beta_1+\beta_2)$, we have

$$\|x_d(t)\|^2 \le 2\|\Theta_d(t, t-\tau) x_d(t-\tau)\|^2$$

$$+ 2\left\|\int_{t-\tau}^t \Theta_d(t,s) B_d [(I + \Delta B_d) u^*(s) + g_d(s, x_d(s))]\, ds\right\|^2$$

$$\le 2c_8 \|x_d(t-\tau)\|^2 + 4c_9 \|B_d\|^2 \Big[(1+\beta_1+\beta_2)^2 \int_{t-\tau}^t \|u^*(s)\|^2\, ds$$

$$+ \gamma^2 \int_{t-\tau}^t \xi^2(y(s))\, ds\Big] \quad (5.53a)$$

where

$$c_8 := \psi_d^2(\tau), \quad c_9 := \int_0^\tau \psi_d^2(s)\, ds \quad (5.53b)$$

Now, from (5.34b) and (5.52) yields

$$y_d(t) = C_d \Theta_d(t, t-\tau) x_d(t-\tau)$$

$$+ C_d \int_{t-\tau}^t \Theta_d(t,s) B_d [(I + \Delta B_d) u^*(s) + g_d(s, x_d(s))]\, ds \quad (5.54)$$

By utilizing (5.49), (5.51) and (5.54),

$$\|x_d(t-\tau)\|^2 \le c_6^{-1}\langle x_d(t-\tau),\ \Lambda_d(t,t-\tau)x_d(t-\tau)\rangle$$

$$= c_6^{-1}\int_{t-\tau}^{t}\|C_d\Theta_d(s,t-\tau)x_d(t-\tau)\|^2\,ds$$

$$= c_6^{-1}\int_{t-\tau}^{t}\|y_d(s) - C_d\int_{t-\tau}^{s}\Theta_d(s,\sigma)B_d[(I+\Delta B_d)u^*(s)$$

$$+ g_d(s,x_d(s))]\,d\sigma\|^2\,ds$$

$$\le 2c_6^{-1}\Big[\int_{t-\tau}^{t}\|y_d(s)\|^2\,ds$$

$$+ 2c_{10}\tau\|C_d\|^2\|B_d\|^2[(1+\beta_1+\beta_2)^2\int_{t-\tau}^{t}\|u^*(s)\|^2\,ds$$

$$+ \gamma^2\int_{t-\tau}^{t}\xi^2(y(s))\,ds]\Big] \tag{5.55a}$$

where

$$c_{10} := \int_0^{\tau}\int_0^{s}\psi_d^2(\sigma)\,d\sigma\,ds, \tag{5.55b}$$

and we use the fact (since $s \in [t-\tau, t]$) that

$$\int_{t-\tau}^{s}\|u^*(\sigma)\|^2\,d\sigma \le \int_{t-\tau}^{t}\|u^*(\sigma)\|^2\,d\sigma,$$

and

$$\int_{t-\tau}^{s}\xi^2(y(\sigma))\,d\sigma \le \int_{t-\tau}^{t}\xi^2(y(\sigma))\,d\sigma.$$

Combining (5.53) and (5.55) yields the result.

We are now ready to state and prove the stability theorem of the adaptively controlled differential inclusion system (5.38).

Theorem 5.3

For all initial data $(t_0, x_d(t_0), \kappa_d(t_0)) \in \mathbb{R}\times\mathbb{R}^{n+\bar{q}}\times(0,\infty)$, the adaptively controlled differential inclusion system (5.38) possesses the following properties:

(i) there exists a solution on $[t_0, \omega_1)$, and every such solution can be extended into a solution on $[t_0, \infty)$;

(ii) $\lim_{t\to\infty} \kappa_d(t)$ exists and is finite;

(iii) $\lim_{t\to\infty} \|x_d(t)\| = 0$.

Proof

Multifunction $\mathbf{F}_d$ defined by (5.37) is upper semi-continuous with convex and compact values in $\mathbb{R}\times\mathbb{R}^{n+\bar{q}}\times(0,\infty)$. Thus, for each $(t_0, x_d(t_0), \kappa_d(t_0)) \in \mathbb{R}\times\mathbb{R}^{n+\bar{q}}\times(0,\infty)$, there exists a local solution (x_d, κ_d): $[t_0, \omega_1) \to \mathbb{R}^{n+\bar{q}}\times(0,\infty)$. It remains to show that $\omega_1 = \infty$. We will show this by several steps. First, we prove that κ_d is bounded on $[t_0, \omega_1)$.

Now, seeking a contradiction to above, i.e. suppose that the monotonically increasing function $t \mapsto \kappa_d(t)$ is unbounded. Then, for some $t_1 \in [0, \omega_1)$, $t_1 < \omega_1$, $\kappa_d(t_0 + t_1) = \hat{\kappa}_d > \max\{\kappa_d^*, (1-\beta_2)^{-1}\gamma\}$ and $(\varepsilon\kappa_d(t_0 + t_1))^{-1} = \mu_d < \mu_d^*$. Hence, by using arguments similar to those used in the proof of Theorem 4.4, it can be established that $x(\cdot)$ (and hence $y(\cdot) = Cx(\cdot)$) is ultimately exponentially decaying on $[t_0, \omega_1)$ (and hence are square integrable on $[t_0, \omega_1)$). By continuity of ξ and the exponential decay of y, $(\xi \circ y)(\cdot)$ is bounded and hence $\xi^2(y(\cdot))$ is bounded. Now, consider the filter equation part of (5.34c), i.e.

$$\dot{z}(t) = \varepsilon\kappa_d(t)[\mathcal{A}z(t) + \mathcal{B}y(t)] \tag{5.56}$$

Let φ_d (with inverse φ_d^{-1}) be the monotonic function $t \mapsto \int_{t_0}^{t} \varepsilon\kappa_d(s)\,ds$. Then, it can be verified that

$$z(t) = \exp\,(\mathcal{A}\varphi_d(t))z(t_0) + \int_0^{\varphi_d(t)} \exp\,[\mathcal{A}(\varphi_d(t) - s)]\mathcal{B}y(\varphi_d^{-1}(s))\,ds \quad (5.57)$$

satisfies (5.56). Since $y(\cdot)$ is exponentially decaying, $y(\varphi_d^{-1}(\cdot))$ is clearly bounded. In view of $\sigma(\mathcal{A}) \subset \mathbb{C}^-$, we may conclude from (5.57) that z is bounded. Hence, from (5.34d), $\dot{\kappa}_d(t)$ is bounded and so there exists a constant κ_5 such that

$$\kappa_d(t) \le \kappa_d(t_0) + \kappa_5(t - t_0), \ \text{ for all } t \ge t_0 \ . \quad (5.58)$$

Now, it can be shown that the function $y(\varphi_d^{-1}(\cdot))$ ultimately satisfies

$$\|y(\varphi_d^{-1}(s))\| \le \kappa_6 \exp\,[\kappa_7 - \sqrt{(\kappa_7^2 + \kappa_8 s)}] \quad (5.59)$$

for some positive constants κ_i $(i = 6,7,8)$, and so is square integrable on $[t_0, \omega_1)$. Again, since $\sigma(\mathcal{A}) \subset \mathbb{C}^-$, we may conclude from (5.57) that $z(\cdot)$ is square integrable on $[t_0, \omega_1)$. Thus, $y_d(\cdot)$ is square integrable on $[t_0, \omega_1)$ which, from (5.34d) and in view of A5.2(v) (i.e. $(\xi \circ y)(\cdot)$ is square integrable on $[t_0, \infty)$), contradicts our supposition that κ_d is unbounded. This establishes that $\kappa_d(\cdot)$ is bounded on $[t_0, \omega_1)$.

Secondly, we show that $x_d(\cdot)$ is bounded on $[t_0, \omega_1)$. Let $u^*(\cdot)$ be as in (5.41) and initially we want to estimate $u^*(\cdot)$. Since $\kappa_d(\cdot)$ is bounded, $\kappa_d(t) \in \Omega$, for all t, where Ω is a compact set. Since $\kappa_d \mapsto \|K_d(\kappa_d)\|$ is continuous, then

$$\gamma_d := \max\,\{\|K_d(\kappa_d)\| : \kappa_d \in \Omega\} \quad (5.60)$$

exists. Hence, $\|K_d(\kappa_d(t))\| \le \gamma_d$, for all t. Thus, it follows from (5.45) that,

$$\|u^*(t)\|^2 \le 2\kappa_\infty^2\,[\gamma_d^2 \|y_d(t)\|^2 + \xi^2(y(s))]\,, \ \ \kappa_\infty < \infty\,. \quad (5.61)$$

Now by using (5.61) in Lemma 5.4, yields

$$\|x_d(t)\|^2 \le c_d \int_{t-\tau}^{t} [\|y_d(s)\|^2 + \|u^*(s)\|^2 + \xi^2(y(s))]\, ds$$

$$\le c_{11} \int_{t-\tau}^{t} \|y_d(s)\|^2\, ds + c_{12} \int_{t-\tau}^{t} \xi^2(y(s))\, ds \tag{5.62a}$$

where

$$c_{11} := c_d(1 + 2\kappa_\infty^2 \gamma_d^2), \quad c_{12} := c_d(1 + 2\kappa_\infty^2)\,. \tag{5.62b}$$

In view of A5.2(v), $(\xi \circ y)(\cdot)$ is square integrable on $[t_0, \infty)$ and since $y_d(\cdot)$ is square integrable on $[t_0, \omega_1)$, then we may conclude that $x_d(\cdot)$ is bounded on $[t_0, \omega_1)$.

We have now shown that $(x_d(\cdot), \kappa_d(\cdot))$ is bounded on $[t_0, \omega_1)$. Thus, it follows that every such solution $(x_d, \kappa_d)\colon [t_0, \omega_1) \to \mathbb{R}^{n+\bar{q}} \times (0, \infty)$ with initial value $(t_0, x_d(t_0), \kappa(t_0))$, evolves within a compact set, and hence can be extended indefinitely, i.e. $\omega_1 = \infty$, which proves assertion (i). Furthermore, in view of above arguments, assertion (ii) of the theorem follows.

It remains to show assertion (iii) of the theorem, i.e. $x_d(t) \to 0$ as $t \to \infty$. Clearly, (ii) ensures that $y_d(\cdot)$ is square integrable on $[t_0, \infty)$. We claim that $u^*(\cdot)$ defined as in (5.41) is also square integrable on $[t_0, \infty)$. This, can be easily seen by integrating (5.61) from t_0 to ∞ which yields

$$\int_{t_0}^{\infty} \|u^*(s)\|^2\, ds \le c_{13} \int_{t_0}^{\infty} \|y_d(s)\|^2\, ds + c_{14} \int_{t_0}^{\infty} \xi^2(y(s))\, ds \tag{5.63a}$$

where

$$c_{13} := 2\kappa_\infty^2 \gamma_d^2\,, \quad c_{14} := 2\kappa_\infty^2\,. \tag{5.63b}$$

In view of A5.2(v) and since y_d is square integrable on $[t_0, \infty)$, we conclude that

$$\int_{t_0}^{\infty} \|u^*(s)\|^2\, ds \; (= \lim_{t \to \infty} \int_{t_0}^{t} \|u^*(s)\|^2\, ds)$$

exists and is finite, which establishes our claim. Now, using Lemma 5.4 with $u^*(\cdot)$ as in (5.41), we have

$$\|x_d(t)\|^2 \le c_d \int_{t-\tau}^{t} [\|y_d(s)\|^2 + \|u^*(s)\|^2 + \xi^2(y(s))]\, ds$$

$$= c_d \int_{t_0}^{t} [\|y_d(s)\|^2 + \|u^*(s)\|^2 + \xi^2(y(s))]\, ds$$

$$- c_d \int_{t_0}^{t-\tau} [\|y_d(s)\|^2 + \|u^*(s)\|^2 + \xi^2(y(s))]\, ds \qquad (5.64)$$

Since $\int_{t_0}^{\infty} \gamma_i(s)\, ds$ $(= \lim_{t\to\infty} \int_{t_0}^{t} \gamma_i(s)\, ds)$ is finite, where

$$\gamma_1(\cdot) = \|y_d(\cdot)\|^2,$$

$$\gamma_2(\cdot) = \|u^*(\cdot)\|^2,$$

$$\gamma_3(\cdot) = \xi^2(y(\cdot)),$$

then $\lim_{t\to\infty} \int_{t_0}^{t-\tau} \gamma_i(s)\, ds$ is also finite and equals $\int_{t_0}^{\infty} \gamma_i(s)\, ds$. Hence, $\|x_d(t)\| \to 0$ as $t \to \infty$.

5.5 An example - Suspension control system for a Maglev vehicle

In this section, we give a magnetic levitation (Maglev) vehicle example to illustrate the application of the proposed control described in § 5.3.2. Specifically, the point mass model of Breinl and Leitmann (1983) (see also, Ryan and Corless 1984 and Chen 1986a, b) is adopted, and the same numerical values are used here. We consider only the vertical motion of a single support magnet (Fig. 5.3), where the system without control is unstable.

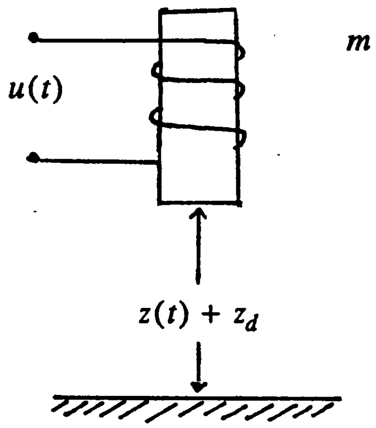

Figure 5.3.

In state space it is governed by

$$\dot{x}(t) = [A + \Delta A_r(t)]x(t) + [B + \Delta B_r(t)]u(t), \quad x(t_0) = x_0 \,, \tag{5.65}$$

where

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix}; \; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}; \tag{5.66a}$$

with

$$a_1 := \frac{K_1 R}{mL_0}, \; a_2 := \frac{K_1}{m} - \frac{K_2 K_3}{mL_0}, \; a_3 := -\frac{R}{L_0}, \; b := -\frac{K_2}{mL_0}, \tag{5.66b}$$

where m is the mass of the magnet; R is resistance; K_1, K_2 and K_3 are gap, current and velocity coefficients, respectively; and L_0 is the nominal inductance. The state vector $x(t) \in \mathbb{R}^3$ consists of the gap deviation $z(t)$ with respect to the desired gap width z_d, velocity $\dot{z}(t)$ and acceleration $\ddot{z}(t)$. The (scalar) control $u(t) \in \mathbb{R}$ is the deviation (from nominal) of applied voltage generating the magnetic field. Furthermore, we assumed that the input disturbances, e.g. due to track irregularities are neglected.

In practice, it is very difficult to measure the inductance accurately. Thus, the inductance considered as uncertain, gives rise to the uncertain elements $\Delta A_r(\cdot)$ and $\Delta B_r(\cdot)$ in the model. Particularly,

$$\Delta A_r(t) = L_r(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_1 & \hat{a}_2 & -a_3 \end{bmatrix}, \quad \hat{a}_2 := \frac{K_2 K_3}{m L_0} \tag{5.67}$$

and

$$\Delta B_r(t) = -L_r(t) B , \tag{5.68}$$

where the uncertain parameter $L_r(t)$ represents the ratio of inductance error $L(t) - L_0$ to actual inductance $L(t)$, i.e.

$$L_r(t) = \frac{L(t) - L_0}{L(t)} , \tag{5.69}$$

and is assumed bounded, i.e.

$$|L_r(t)| \le L_r^* < 1 \tag{5.70}$$

where L_r^* is a known constant (which plays the role of β). Moreover, the function $L_r : \mathbb{R} \to [-L_r^*, L_r^*]$ is assumed to be continuous.

It is assumed that z and $\ddot{z}$ are available for measurement, thus the output of the system is given by

$$y(t) = Cx(t), \tag{5.71a}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.71b}$$

Let us now check that all assumptions of design are hold.

(i) (A, B) is controllable (obvious). Thus, A4.1 holds.

(ii) Here we use $r = 2$ so that $C_r = F_1C + F_2CA$ with $F_1 = [\lambda^2 \;\; 1]$ and $F_2 = [-2\lambda \;\; 0]$, and $\lambda < 0$ is a design parameter.

(a) A4.2(i) holds since $F_2CB = 0$.

(b) With F_1 and F_2 as above, we have

$$C_r = [\lambda^2 \;\; -2\lambda \;\; 1]\,;$$

thus the transfer function of the linear system (C_r, A, B) has the form $G(s) = N(s)D^{-1}(s)$ with $N(s) = (s-\lambda)^2$ and $D(s) = s^3 - a_3s^2 - a_2s - a_1$. Hence, $|N(s)| = 0 \Rightarrow (s-\lambda)^2 = 0 \Rightarrow s = \lambda$ $(s \in \mathbb{C}^-)$. Thus, A4.2(ii) holds.

(c) $C_rB = b \Rightarrow \det(C_rB) = \det b \neq 0$. Hence, A4.2(iii) holds.

(iii) $g(t,x,u) = \Delta A(t)x + g_3(t,u)$,

where

$$\Delta A(t) = [\frac{K_1R}{K_2}L_r \quad -K_3L_r \quad -\frac{mR}{K_2}L_r]\,, \quad g_3(t,u) = -L_r(t)u\,,$$

with

$$\|g_3(t,u)\| \leq L_r^* \,|u|$$

Hence, A5.1(i),(ii) hold.

(iv) It remains to check that $(C, A + B\Delta A(\cdot))$ is uniformly completely observable in the sense of Definition 2.8.

(a) Let $\Phi(\cdot,\cdot)$ be the state transition matrix function generated by $A + B\Delta A(\cdot)$. Now, for some k_1 and ω, we have $\|e^{At}\| \le k_1 e^{\omega t}$ and since $\Delta A(\cdot)$ is bounded, there exists a constant k_2 such that $\|B\Delta A(t)\| \le k_2$. By standard perturbation theory, it can be shown that

$$\|\Phi(t,s)\| \le k_1 e^{(\omega + k_1 k_2)(t-s)} \quad \text{for all } t, s\,,$$

$$= \alpha_5(|t-s|) \quad \text{for all } t, s\,, \tag{5.72a}$$

where

$$\alpha_5 : \sigma \mapsto k_1 e^{(\omega + k_1 k_2)\sigma}\,. \tag{5.72b}$$

Thus, condition (2.13c) of Definition 2.8 holds.

(b) Next, we want to calculate upper bound for $M(t, t-\tau)$, i.e. the observability Gramian for the pair $(C, A + B\Delta A(\cdot))$ which is given by (5.17). Using (5.72),

$$\|M(t,t-\tau)\| \le \int_{t-\tau}^{t} \|C^T C\| \|\Phi(\sigma, t-\tau)\|^2 \, d\sigma$$

$$\le \|C^T C\| \int_{t-\tau}^{t} \alpha_5^2(\tau)\, d\sigma$$

$$= \tau \|C^T C\| \alpha_5^2(\tau) =: \alpha_2(\tau)$$

(c) Finally, we have to show that M is positive definite. Note initially that the state transition matrix function $\Phi(\cdot,\cdot)$ generated by $A + B\Delta A(\cdot)$ satisfies

$$\Phi(t, t-\tau) = \exp(A\tau) + \int_{t-\tau}^{t} \exp(A(t-\sigma)) B\Delta A(\sigma) \Phi(\sigma, t-\tau)\, d\sigma$$

Now

$$y(t) = C\Phi(t, t-\tau) x(t-\tau)$$

Then,

$$\int_{t-\tau}^{t} \|y(\sigma)\|^2 \, d\sigma = \int_{t-\tau}^{t} \langle C\Phi(\sigma, t-\tau)x(t-\tau), C\Phi(\sigma, t-\tau)x(t-\tau) \rangle \, d\sigma$$

$$= \langle x(t-\tau), \int_{t-\tau}^{t} \Phi^T(\sigma, t-\tau)C^T C\Phi(\sigma, t-\tau) \, d\sigma \, x(t-\tau) \rangle$$

$$= \langle x(t-\tau), \mathrm{M}(t, t-\tau)x(t-\tau) \rangle \qquad (5.73)$$

Now, the matrix C can be written as

$$C = C_1^0 + C_2^0$$

where

$$C_1^0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } C_2^0 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$y(t) = Cx(t) = C_1^0 x(t) + C_2^0 x(t)$$

$$= y_1(t) + y_2(t) \qquad (5.74)$$

Therefore, from (5.73) and (5.74),

$$\int_{t-\tau}^{t} \|y(\sigma)\|^2 \, d\sigma = \int_{t-\tau}^{t} \|y_1(\sigma)\|^2 \, d\sigma + \int_{t-\tau}^{t} \|y_2(\sigma)\|^2 \, d\sigma \geq \int_{t-\tau}^{t} \|y_1(\sigma)\|^2 \, d\sigma$$

Assume now $\int_{t-\tau}^{t} \|y_1(\sigma)\|^2 \, d\sigma = 0$. Therefore, $y_1(s) = 0$ for $t-\tau \leq s \leq t$ which implies $C_1^0 x(s) = 0$ and, in particular, $C_1^0 x(t-\tau) = 0$.

Also

$$C_1^0 \dot{x}(s) = C_1^0 [A + B\Delta A(s)]x(s)$$

$$= C_1^0 A x(s) \quad (C_1^0 B \Delta A(s) = 0)$$

$$= 0, \quad t-\tau \leq s \leq t,$$

and, in particular,

$$C_1^0 A x(t-\tau) = 0 \,.$$

Similarly, we have

$$\begin{aligned} C_1^0 \ddot{x}(s) &= C_1^0 A \dot{x}(s) \\ &= C_1^0 A [A + B\Delta A(t)] x(s) \\ &= C_1^0 A^2 x(s) \\ &= 0 \,, \quad t-\tau \le s \le t \,, \end{aligned}$$

and, in particular,

$$C_1^0 A^2 x(t-\tau) = 0 \,.$$

Hence,

$$\begin{bmatrix} C_1^0 \\ C_1^0 A \\ C_1^0 A^2 \end{bmatrix} x(t-\tau) = 0$$

But (C_1^0, A) is an observable pair and so $x(t-\tau) = 0$. Thus,

$$x(t-\tau) \neq 0 \Rightarrow \int_{t-\tau}^{t} \|y_1(\sigma)\|^2 \, d\sigma > 0 \Rightarrow \int_{t-\tau}^{t} \|y(\sigma)\|^2 \, d\sigma > 0 \,.$$

Hence, M is positive definite.

From (b) and (c), the condition (2.13a) of Definition 2.8 holds. Consequently, we can conclude from (a), (b) and (c) that $(C, A + B\Delta A(\cdot))$ is uniformly completely observable.

Now, for simulation we return to equation (5.3) with the adaptation law (5.12). For this example, the filter dynamic (i.e. equation (5.3b)) is a scalar (to estimate x_2). A realization of the filter dynamic has the form (in terms of the state variables)

$$\dot{x}_f(t) = -\delta\kappa(t)[x_f(t) - 2\lambda x_1(t)] \tag{5.75}$$

with the output

$$z(t) = \delta\kappa(t)[x_f(t) - 2\lambda x_1(t)] \tag{5.76}$$

wherein we have replaced fixed μ in (5.3b) by variable $(\delta\kappa(t))^{-1}$, where $\delta > 0$ is a design parameter and $\kappa(t) > 0$ is generated by the adaptation law (5.12), i.e.

$$\dot{\kappa}(t) = x_1^2(t) + x_3^2(t) + x_f^2(t)\,. \tag{5.77}$$

The overall control (equivalent to equation (5.3d)) then is given by

$$u(t) = -\kappa(t)(C_rB)^{-1}\,[\lambda^2 x_1(t) + x_3(t) + \delta\kappa(t)[x_f(t) - 2\lambda x_1(t)]] \tag{5.78}$$

For purposes of simulation, the following illustrative (numerical) parameters are adopted (Breinl and Leitmann 1983):

$$m = 16\,\text{kg},\ \ R = 8\,\Omega,\ \ K_1 = 5.7\times10^4\,\text{N m}^{-1},\ \ K_2 = K_3 = 114\,\text{N A}^{-1},$$

$$L_0 = 0.5\,\text{V s A}^{-1},\ \ \text{with}\ \ L_r^* = 0.5\,.$$

The control design parameters used in simulation:

$$\delta = 10\,,\quad \lambda = -15\,.$$

Figures 5.4-5.9 depict the simulated evolution of states, filter's state, adaptation gain and control for an initial value

$$(x_1(0), x_2(0), x_3(0), x_f(0), \kappa(0)) = (10^{-3}, 5\times10^{-3}, 0, 0, 0.1)\,.$$

It is clearly seen from simulations that the example illustrated the proposed control design.

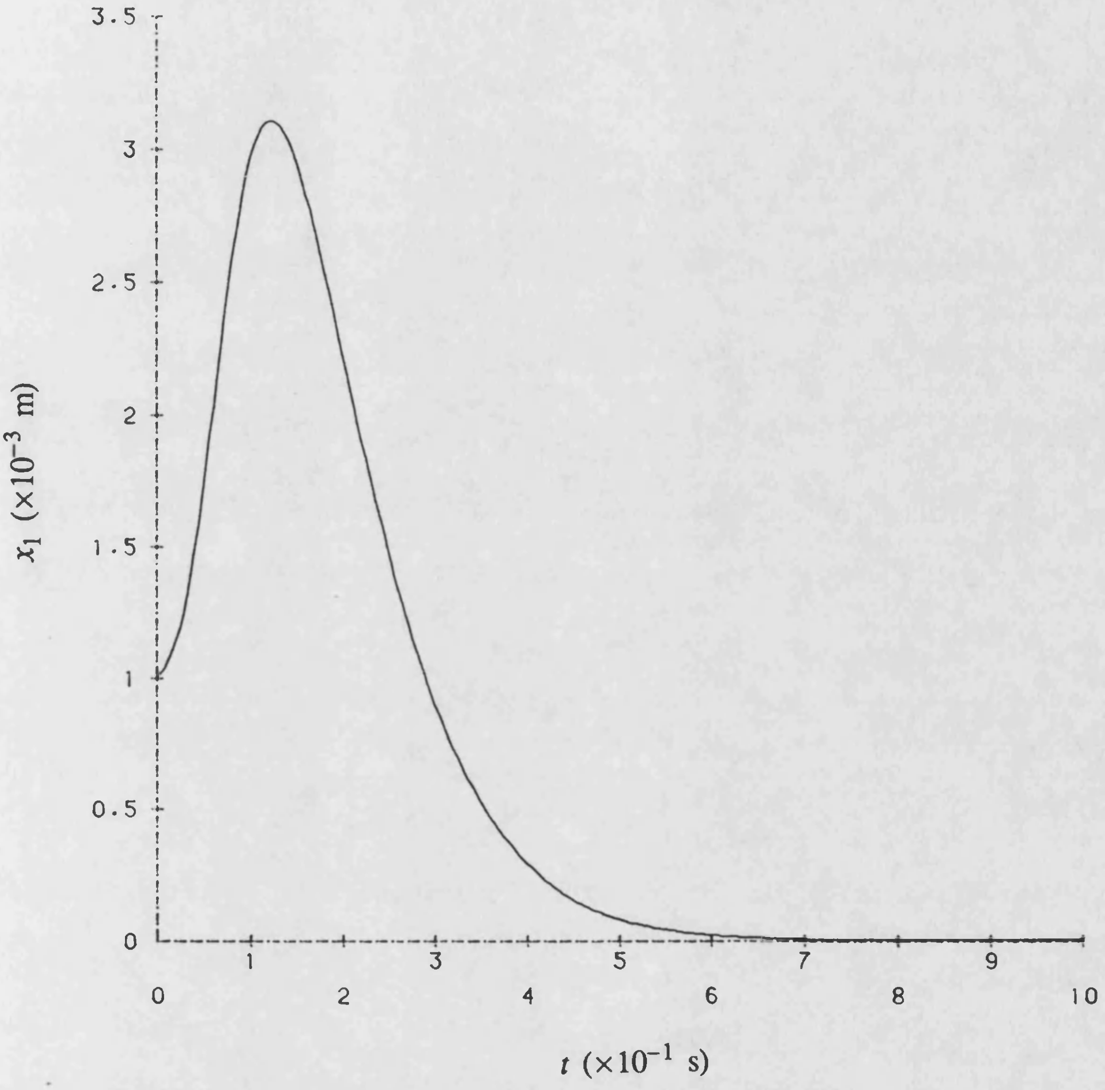

Figure 5.4. Evolution of state x_1

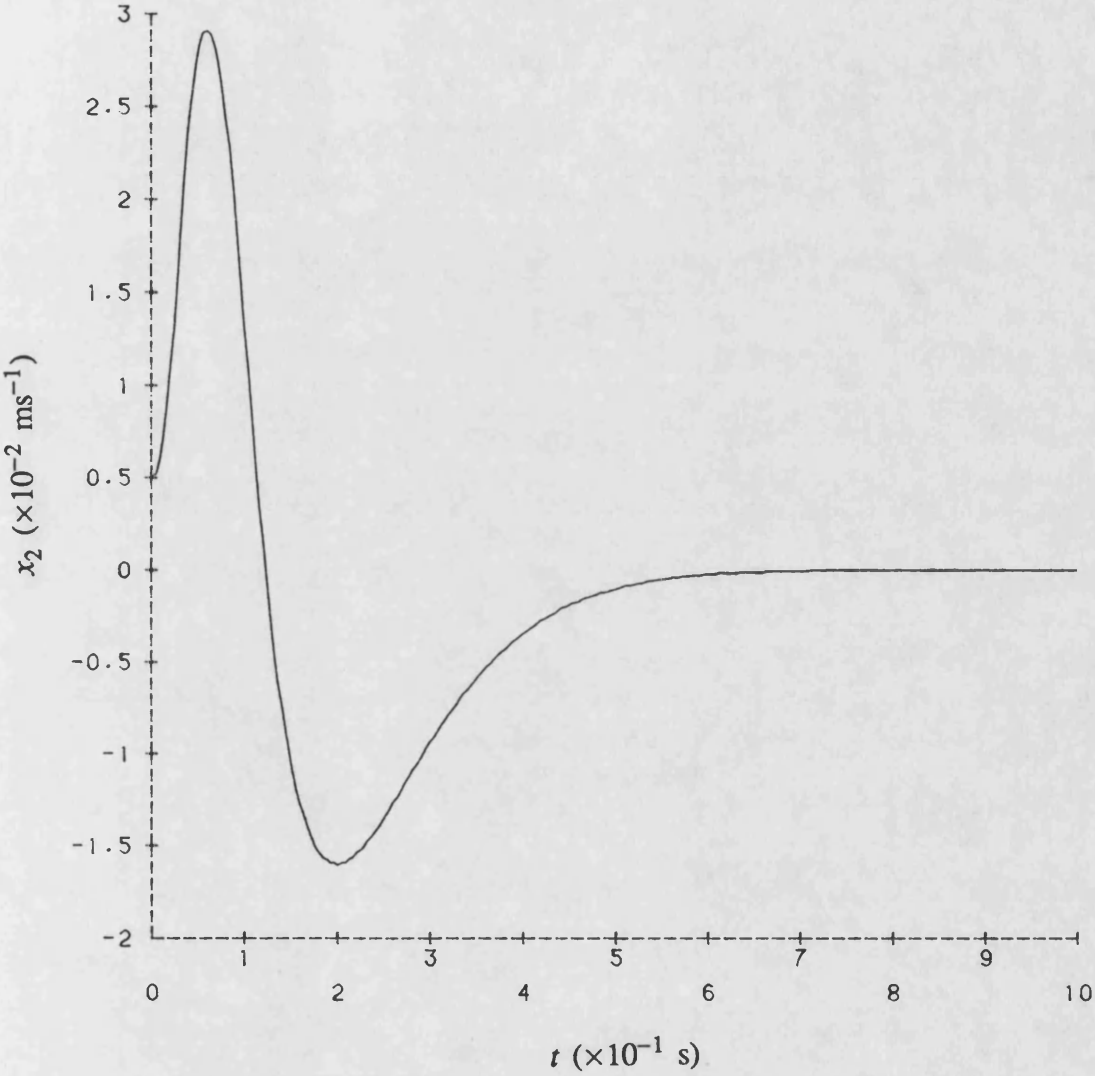

Figure 5.5. Evolution of state x_2

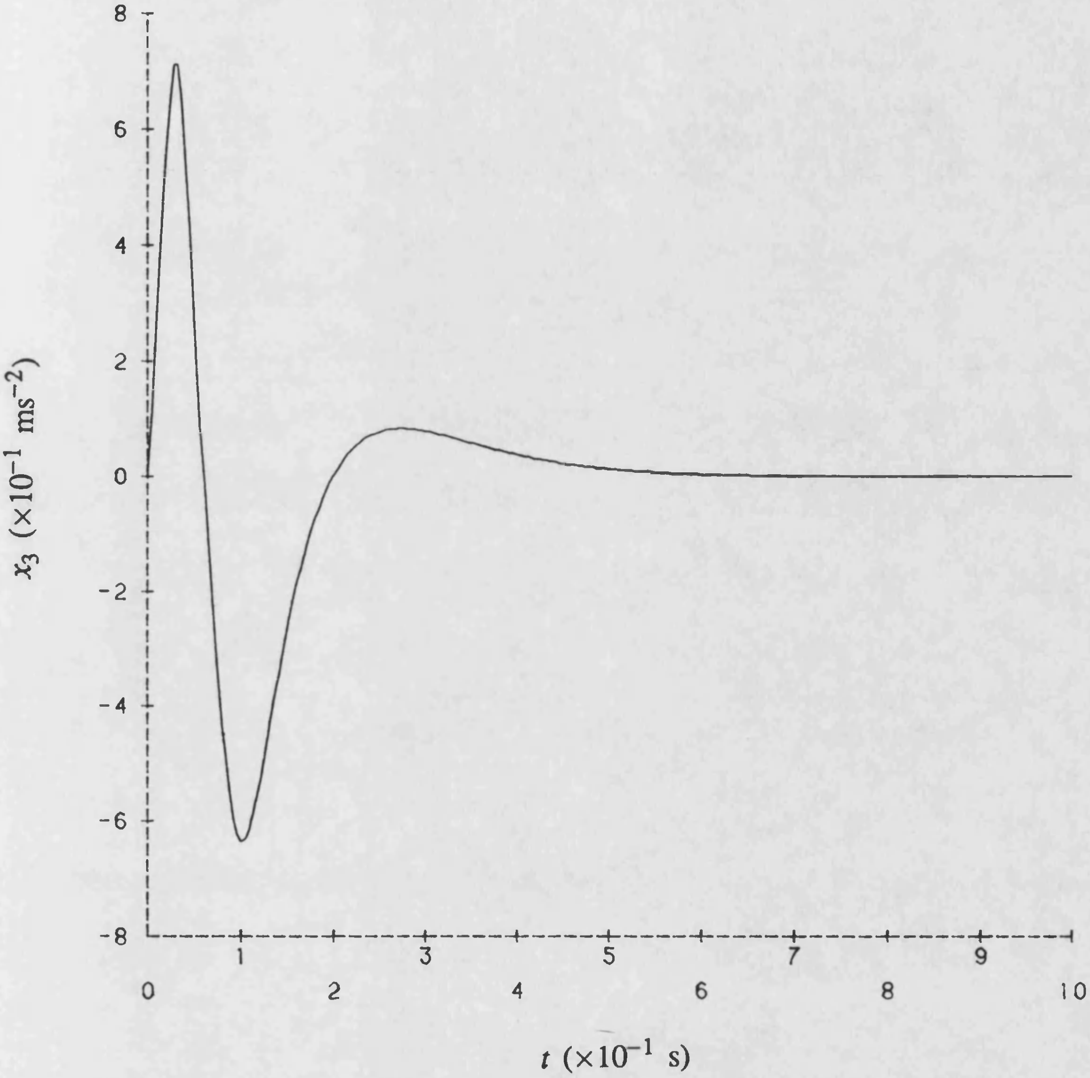

Figure 5.6. Evolution of state x_3

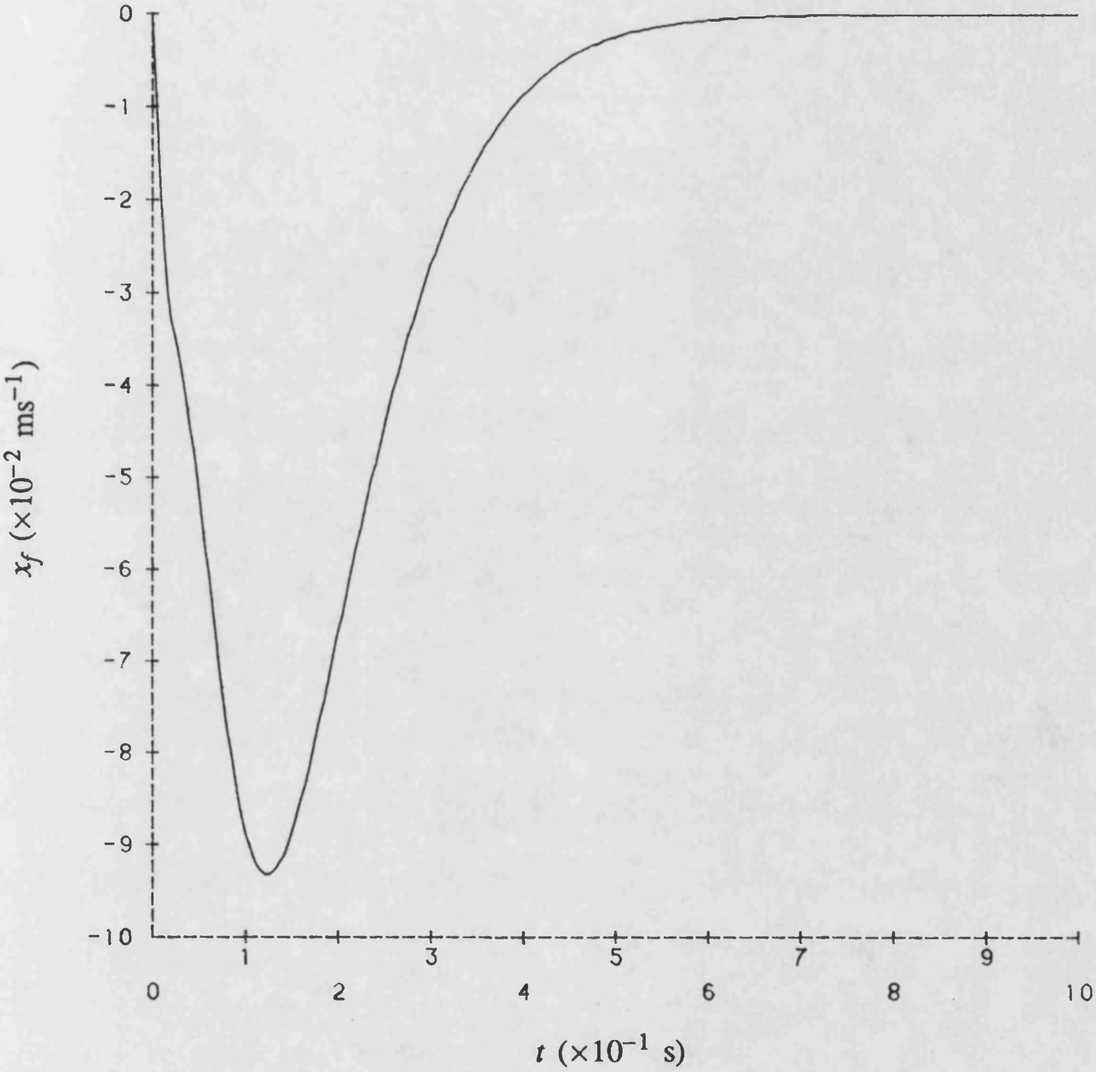

Figure 5.7. Evolution of filter's state x_f

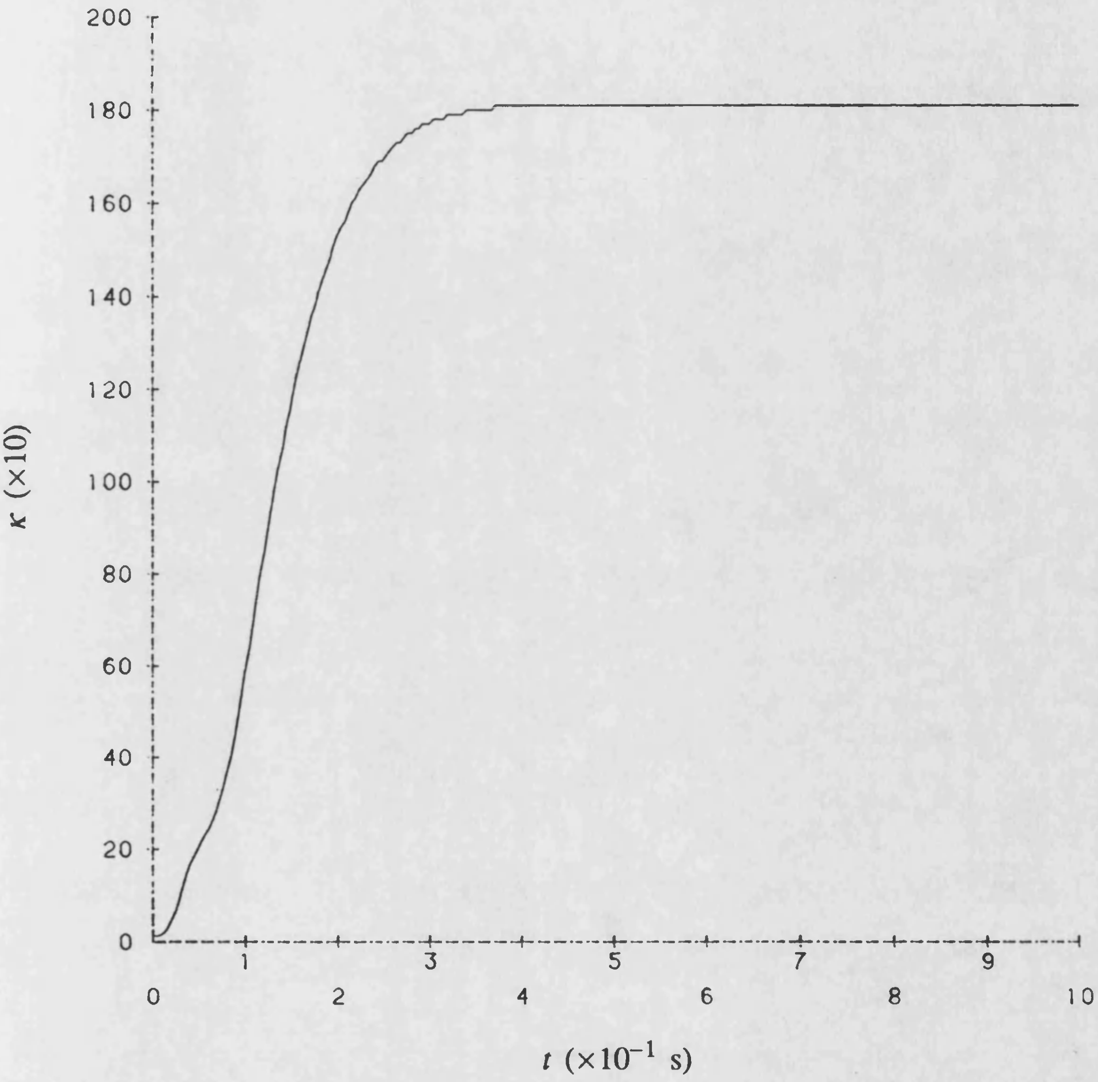

Figure 5.8. Evolution of adaptation gain κ

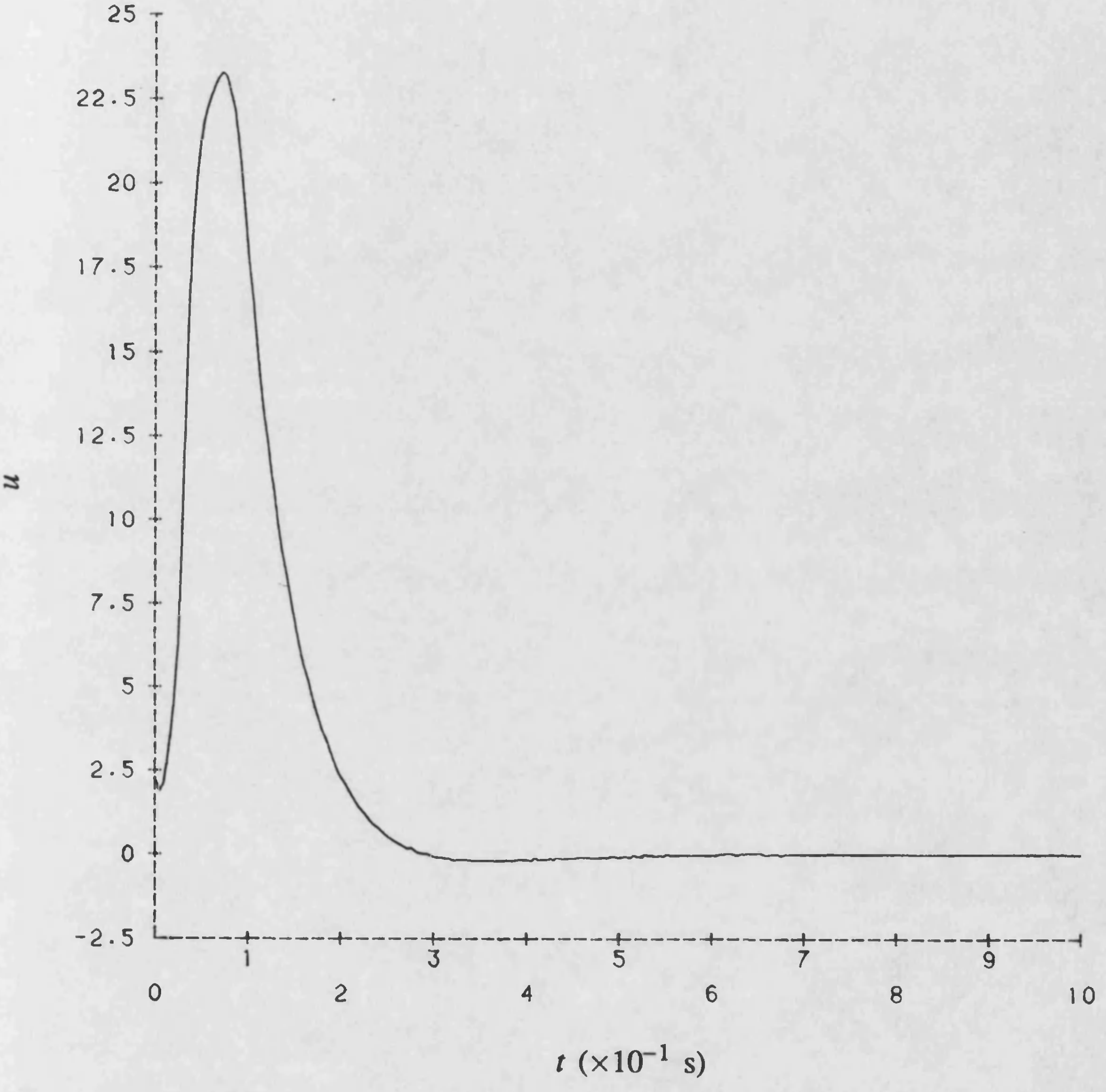

Figure 5.9. Evolution of control

CHAPTER 6

STATIC OUTPUT FEEDBACK STABILIZATION FOR A CLASS OF UNCERTAIN "RELATIVE DEGREE TWO" SYSTEMS

6.1 Introduction

In this chapter, we address the problem of designing static output feedback control for a class of uncertain "relative degree 2" systems. The approach is anologous to that of Chapter 4, but with a fundamental distinction: in Chapter 4, a realizable dynamic compensator is used to stabilize a class of uncertain systems; in this chapter, a class of uncertain systems is stabilized by using only a static output feedback control.

To achieve our aim, we have to impose an extra or additional set of assumptions to the system. It is shown that, a cone-bounded uncertainty can be tolerated by a static output feedback. Since the feedback control is based on "worst case" design, the proposed feedback control is expected to be conservative. Thus, anologous to Chapter 5, an adaptive version of this feedback control is conjectured to allow for bounded uncertainties with unknown bounds and to counteract conservatism.

The chapter is presented as follows. In § 6.2, we first state the system and impose a set of assumptions which implicitly defined the class of systems to be studied. Then, by using an approach anologous to that of Chapter 4, we establish the existence of a class of stabilizing static output feedback control for the system. Finally, in § 6.4, an adaptive version is conjectured (anologous to

Chapter 5) which may counteract the conservatism that induced by crude estimates in the "worst case" design and which also may dispense with the requirement that uncertainty parameters be known.

6.2 The system and assumptions

The system to be considered is of the form

$$\dot{x}(t) = Ax(t) + B[u(t) + g(t, x(t), u(t))], \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \tag{6.1}$$

with an output given by

$$y(t) = Cx(t), \quad y(t) \in \mathbb{R}^m. \tag{6.2}$$

First, we impose assumptions on the nominal linear system (C, A, B).

A6.1: (i) Transmission zeros of (C, A, B) lie in $\mathbb{C}^-$;

(ii) $CB = 0$;

(iii) CAB is nonsingular;

(iv) Spectrum of $CA^2B(CAB)^{-1}$ lies in $\mathbb{C}^-$.

Next we impose structural properties on g, which implicitly define the class of uncertain systems to be studied.

A6.2: (i) g: $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function;

(ii) For all $(t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$,

$$g(t, x, u) = g_1(t, x) + g_2(t, Cx) + \gamma u,$$

with

(a) $\|g_1(t,x)\| \le \alpha_1 \|x\|$, where α_1 is a known constant;

(b) $\|g_2(t,y)\| \le \alpha_2 \|y\|$, where α_2 is a known constant;

(c) there exists γ^* such that $|\gamma| \le \gamma^* < 1$.

6.3 Stabilizing static output feedback

In this section, we consider the problem of designing of static output feedback control for the class of systems described in the previous section. In order to proceed, we first introduce the following notation and state transformation.

Let

$$\bar{B} = [B \,\vdots\, AB] \text{ and } \bar{C} = \begin{bmatrix} C \\ CA \end{bmatrix} \tag{6.3}$$

Then, by straightforward calculation,

$$\bar{C}\bar{B} = \begin{bmatrix} 0 & CAB \\ CAB & CA^2B \end{bmatrix} \tag{6.4a}$$

and

$$(\bar{C}\bar{B})^{-1} = \begin{bmatrix} -(CAB)^{-1}CA^2B(CAB)^{-1} & (CAB)^{-1} \\ (CAB)^{-1} & 0 \end{bmatrix} \tag{6.4b}$$

Now, let $T \in \mathbb{R}^{(n-2m)\times n}$ be such that $\ker T = \operatorname{im} \bar{B}$, then $\bar{T} := \begin{bmatrix} T \\ \bar{C} \end{bmatrix}$ is invertible, with inverse $\bar{S} = [S \,\vdots\, \bar{B}(\bar{C}\bar{B})^{-1}]$, where

$$S := (I - \bar{B}(\bar{C}\bar{B})^{-1}\bar{C})T^T(TT^T)^{-1}. \tag{6.5}$$

For convenience, we write

$$M = CA^2B(CAB)^{-1} \tag{6.6}$$

We now introduce the coordinate transformation (parameterized by $k > 0$)

$$x \mapsto L_k x = \tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{6.7a}$$

where

$$L_k := \begin{bmatrix} T \\ kC \\ CA - \tfrac{1}{2}MC \end{bmatrix} \tag{6.7b}$$

with inverse

$$L_k^{-1} = [S \,\vdots\, k^{-1}S_1 \,\vdots\, B(CAB)^{-1}] \tag{6.7c}$$

where

$$S_1 := AB(CAB)^{-1} - \tfrac{1}{2}B(CAB)^{-1}M \tag{6.7d}$$

In new coordinates the system representation is

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}[u(t) + \tilde{g}(t, \tilde{x}(t), u(t))] \tag{6.8a}$$

with output

$$y(t) = \tilde{C}\tilde{x}(t) \tag{6.8b}$$

where

$$\tilde{A} = L_k A L_k^{-1} = \begin{bmatrix} A^* & k^{-1}A_1 & 0 \\ 0 & \tfrac{1}{2}M & kI \\ A_2 & k^{-1}A_3 & \tfrac{1}{2}M \end{bmatrix}, \tag{6.8c}$$

$$\tilde{B} = L_k B = \begin{bmatrix} 0 \\ 0 \\ CAB \end{bmatrix}, \tag{6.8d}$$

$$\tilde{C} = CL_k^{-1} = [0 \,\vdots\, k^{-1}I \,\vdots\, 0], \tag{6.8e}$$

with

$$A^* = TAS\,, \quad A_1 = TA^2B(CAB)^{-1}\,,$$
$$A_2 = CA^2S\,, \quad A_3 = CA^3B(CAB)^{-1} - \tfrac{3}{4}M^2\,, \tag{6.8f}$$

and

$$\tilde{g}(t,\tilde{x},u) := g(t,L_k^{-1}\tilde{x},u)\,. \tag{6.8g}$$

We now introduce the output feedback

$$u(t) = -k^2(CAB)^{-1}y(t) = -k(CAB)^{-1}x_2(t)\,. \tag{6.9}$$

Then, the closed-loop feedback system now becomes

$$\dot{x}_1(t) = A^*x_1(t) + k^{-1}A_1x_2(t) \tag{6.10a}$$

$$\dot{x}_2(t) = \tfrac{1}{2}Mx_2(t) + kx_3(t) \tag{6.10b}$$

$$\dot{x}_3(t) = A_2x_1(t) + k^{-1}A_3x_2(t) + \tfrac{1}{2}Mx_3(t) - kx_2(t)$$
$$+ (CAB)\tilde{g}(t,\tilde{x}(t),-k(CAB)^{-1}x_2(t)) \tag{6.10c}$$

In view of A6.1(i), $\sigma(A^*) \subset \mathbb{C}^-$ and hence

$$P^*A^* + (A^*)^TP^* + I = 0 \tag{6.11}$$

has unique symmetric positive definite solution P^*. Also, in view of A6.1(iv), $\sigma(M) \subset \mathbb{C}^-$ and hence

$$PM + M^TP + I = 0 \tag{6.12}$$

has unique symmetric positive definite solution P.

We now impose our final assumption.

A6.3: $\quad \alpha_1 < \dfrac{1}{4\|P\|\|CAB\|\|B(CAB)^{-1}\|}$

Regarding the feedback controlled system (6.10), we have

Theorem 6.1

There exists $k^* \in \mathbb{R}$ such that, for each fixed $k > k^*$ the feedback controlled system (6.10) is globally uniformly asymptotically stable.

Proof

The Carathéodory assumption (A6.2(i)) on g ensures that, for each $(t_0, \tilde{x}_0) \in \mathbb{R}\times\mathbb{R}^n$ there exists a local solution $\tilde{x}(\cdot)$ of (6.10) with $\tilde{x}(t_0) = \tilde{x}_0$.

Introduce Lyapunov function candidate $V_k\colon \mathbb{R}^{n-2m}\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}$ defined by

$$V_k(x_1, x_2, x_3) := \tfrac{1}{2}k\langle x_1, P^* x_1\rangle + (1+\gamma)\langle x_2, Px_2\rangle + \langle x_3, Px_3\rangle \tag{6.13}$$

Then, along every trajectory $(x_1(\cdot), x_2(\cdot), x_3(\cdot))$ of (6.10), the following holds almost everywhere

$$\begin{aligned}\frac{d}{dt}V_k(x_1(t), x_2(t), x_3(t)) = {} & k\langle P^* x_1(t), A^* x_1(t) + k^{-1}A_1 x_2(t)\rangle \\ & + 2(1+\gamma)\langle Px_2(t), \tfrac{1}{2}Mx_2(t) + kx_3(t)\rangle \\ & + 2\langle Px_3(t), A_2 x_1(t) + k^{-1}A_3 x_2(t) + \tfrac{1}{2}Mx_3(t) \\ & - kx_2(t) + (CAB)\tilde{g}(t, \tilde{x}(t), -k(CAB)^{-1}x_2(t))\rangle\end{aligned}$$

In view of A6.2, (6.7c) and (6.8g),

$$\frac{d}{dt}V_k(x_1(t),x_2(t),x_3(t)) \le -\tfrac{1}{2}k\|x_1(t)\|^2 + \|P^*A_1\|\|x_1(t)\|\|x_2(t)\|$$

$$-\tfrac{1}{2}(1+\gamma)\|x_2(t)\|^2 + 2k(1+\gamma)\langle Px_2(t),x_3(t)\rangle$$

$$+2\|PA_2\|\|x_1(t)\|\|x_3(t)\| + 2k^{-1}\|PA_3\|\|x_2(t)\|\|x_3(t)\|$$

$$-\tfrac{1}{2}\|x_3(t)\|^2 - 2k\langle Px_3(t),x_2(t)\rangle$$

$$+2\|P\|\|CAB\|\,[\alpha_1\|S\|\|x_1(t)\|\|x_3(t)\|$$

$$+\alpha_1k^{-1}\|S_1\|\|x_2(t)\|\|x_3(t)\| + \alpha_1\|B(CAB)^{-1}\|\|x_3(t)\|^2$$

$$+\alpha_2k^{-1}\|x_2(t)\|\|x_3(t)\|] - 2k\gamma\langle Px_3(t),x_2(t)\rangle$$

$$\le -\tfrac{1}{2}\left\langle \begin{bmatrix}\|x_1(t)\|\\ \|x_2(t)\|\\ \|x_3(t)\|\end{bmatrix}, M_k\begin{bmatrix}\|x_1(t)\|\\ \|x_2(t)\|\\ \|x_3(t)\|\end{bmatrix}\right\rangle \tag{6.14a}$$

where

$$M_k := \begin{bmatrix} k & -m_1 & -m_2\\ -m_1 & (1-\gamma^*) & -m_3\\ -m_2 & -m_3 & m_4\end{bmatrix} \tag{6.14b}$$

with

$$m_1 = \|P^*A_1\|,\quad m_2 = 2[\|PA_2\| + \alpha_1\|P\|\|CAB\|\|S\|],$$

$$m_3 = 2k^{-1}[\|PA_3\| + \|P\|\|CAB\|(\alpha_1\|S_1\| + \alpha_2)],$$

$$m_4 = 1 - 4\alpha_1\|P\|\|CAB\|\|B(CAB)^{-1}\|.$$

Note that $(1-\gamma^*)$ and m_4 are positive by virtue of A6.2(c) and A6.3. Thus, there exists k^* such that (6.14a) is a positive definite quadratic form for each fixed $k > k^*$. Hence, the result follows.

6.4 Conjectured stabilizing adaptive output feedback

In the previous section, if A6.1-A6.3 hold, the original system (6.1,6.2) is uniformly asymptotically stabilized by the static output feedback (6.9) for each fixed $k > k^*$ and sufficient information is available to compute k^*. Here, we consider the case for which A6.1 holds but now we only require knowledge of CAB. A6.2 and A6.3 also remain in force but the constants α_1 and α_2 in A6.2(ii)(a-b) may be unknown.

Replace fixed $k > k^*$ in (6.9) by variable $\kappa(t)$ to yield

$$u(t) = -\kappa^2(t)(CAB)^{-1}y(t) \tag{6.15}$$

and let $\kappa(t)$ evolve according to an adaptation law

$$\dot{\kappa}(t) = \|(CAB)^{-1}y(t)\|^2 \tag{6.16}$$

Then the adaptively controlled system becomes

$$\dot{x}_1(t) = A^*x_1(t) + k^{-1}A_1x_2(t) \tag{6.17a}$$

$$\dot{x}_2(t) = \tfrac{1}{2}Mx_2(t) + kx_3(t) \tag{6.17b}$$

$$\dot{x}_3(t) = A_2x_1(t) + k^{-1}A_3x_2(t) + \tfrac{1}{2}Mx_3(t) - k^{-1}\kappa^2(t)x_2(t)$$

$$+ (CAB)\tilde{g}(t,\tilde{x},-k^{-1}\kappa^2(t)x_2(t)) \tag{6.17c}$$

$$\dot{\kappa}(t) = \|(CAB)^{-1}y(t)\|^2 \tag{6.17d}$$

We conjecture the following:

Conjecture 6.1

For all initial data $(t_0, \tilde{x}(t_0), \kappa(t_0)) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^+$, the adaptively controlled system (6.17) possesses the following properties:

(i) $\lim_{t \to \infty} \kappa(t)$ exists and is finite;

(ii) $\lim_{t \to \infty} \|\tilde{x}(t)\| = 0$.

A possible proof might be constructed along the following lines.

(i) Suppose that the monotonically increasing function $t \mapsto \kappa(t)$ is unbounded. Then, for some $t_1 \geq 0$, $\kappa(t_0 + t_1) = k > k^*$. Hence, the result of Theorem 6.1 would suggest that $\tilde{x}(\cdot)$ (and hence $y(\cdot) = \tilde{C}\tilde{x}(\cdot)$) is ultimately exponentially decaying on $[t_0, \infty)$ (and hence are square integrable on $[t_0, \infty)$). At present, we are unable to prove this. However, if this is true, then x_2 is square integrable on $[t_0, \infty)$, which from (6.17d) contradicts with supposition that $\kappa(t)$ is unbounded. Hence, the results of (i) would follow.

(ii) Now, if $\kappa(t)$ is bounded (say κ_∞), then by virtue of (6.17d), y (and hence x_2) is square integrable on $[t_0, \infty)$. This and in view of asymptotic stability of A^* in (6.17a) yields x_1 (and $\dot{x}_1$) square integrable on $[t_0, \infty)$.

To proceed in the argument, we now consider the subsystem (6.17c) and a Lyapunov function candidate W: $\mathbb{R}^m \to \mathbb{R}$ given by

$$W(x_3) = \langle x_3, Px_3 \rangle \tag{6.18}$$

Then, along any solution $(\tilde{x}(\cdot), \kappa(\cdot))$ of (6.17c), the following holds almost everywhere

$$\frac{d}{dt}W(x_3(t)) = \langle 2Px_3(t), \tfrac{1}{2}Mx_3(t) + A_2x_1(t) + k^{-1}(A_3 - \kappa^2(t)I)x_2(t)$$

$$+ (CAB)\tilde{g}(t,\tilde{x}, -k^{-1}\kappa^2(t)x_2(t))\rangle$$

$$\leq -\tfrac{1}{2}[1 - 4\alpha_1 \|P\|\|CAB\|\|B(CAB)^{-1}\|]\|x_3(t)\|^2$$

$$+ 2\|Px_3(t)\| f(x_1(t), x_2(t)) \tag{6.19a}$$

where

$$f(x_1, x_2) := [\|A_2\| + \alpha_2\|CAB\|\|S_1\|]\|x_1\|$$

$$+ k^{-1}[\|A_3\| + (1-\gamma^*)\kappa_\infty^2 + \|CAB\|(\alpha_1\|S_1\| + \alpha_2)]\|x_2\| \tag{6.19b}$$

Note that, the coefficient of $\|x_3\|^2$ (in the bracket) of (6.19a) is positive by virtue of A6.3 and $f(x_1(\cdot), x_2(\cdot))$ is square integrable on $[t_0, \infty)$ (since x_1 and x_2 are square integrable on $[t_0, \infty)$).

Now, let

$$c := 1 - 4\alpha_1\|P\|\|CAB\|\|B(CAB)^{-1}\| > 0, \tag{6.20}$$

then (6.19a) can be rewritten as

$$\frac{d}{dt}W(x_3(t)) \leq -\tfrac{1}{2}c\|x_3(t)\|^2 + 2\|P\|\|x_3(t)\| f(x_1(t), x_2(t))$$

$$\leq -c_1\|x_3(t)\|^2 + c_2\|x_3(t)\| f(x_1(t), x_2(t)), \quad c_1 = \tfrac{1}{2}c, \quad c_2 = 2\|P\|$$

$$\leq -c_1\|x_3(t)\|^2 - [\alpha\|x_3(t)\| - \frac{c_2}{2\alpha} f(x_1(t), x_2(t))]^2$$

$$+ \alpha^2\|x_3(t)\|^2 + \frac{c_2^2}{4\alpha^2} f^2(x_1(t), x_2(t))$$

$$\leq -\,(c_1 - \alpha^2)\|x_3(t)\|^2 + \frac{c_2^2}{4\alpha^2} f^2(x_1(t), x_2(t)) \tag{6.21}$$

Integrating (6.21) from t_0 to τ, yields

$$W(x_3(\tau)) - W(x_3(t_0)) \leq -(c_1 - \alpha^2)\int_{t_0}^{\tau} \|x_3(t)\|^2\, dt + \frac{c_2^2}{4\alpha^2}\int_{t_0}^{\tau} f^2(x_1(t), x_2(t))\, dt$$

$$\leq -(c_1 - \alpha^2)\int_{t_0}^{\tau} \|x_3(t)\|^2\, dt + \kappa\,,$$

since $f(x_1(\cdot), x_2(\cdot))$ is square integrable on $[t_0, \infty)$. Now, by choosing α such that $c_1 > \alpha^2$ and rearranging, we have, for all $\tau > t_0$,

$$(c_1 - \alpha^2)\int_{t_0}^{\tau} \|x_3(t)\|^2\, dt \leq W(x_3(t_0)) - W(x_3(\tau)) + \kappa$$

$$\leq W(x_3(t_0)) + \kappa = M_1 \tag{6.22}$$

Therefore x_3 is square integrable on $[t_0, \infty)$.

Consider now the subsystem (6.17b). Since $\sigma(M) \subset \mathbb{C}^-$ and subsystem (6.17b) with square integrable input x_3, then x_2 (hence $\dot{x}_2$) is square integrable on $[t_0, \infty)$. Thus, we could conclude that $\tilde{x}$ (and hence $\dot{\tilde{x}}$) is square integrable on $[t_0, \infty)$. Hence, $\|\tilde{x}\| \to 0$ as $t \to \infty$.

CHAPTER 7

CONCLUSIONS

7.1 Introduction

This chapter aims to conclude the thesis by summarizing and discussing the results obtained and briefly indicate some suggestions for future research and highlight some possible extensions and applications.

Main results are summarized and discussed in § 7.2, while in § 7.3 we indicate some possible extensions of our work motivated either by some unresolved problems which arose during the investigation or by potential generalizations to a wider framework.

7.2 Summary and discussion of the main results

In this section, we summarize and discuss the main results obtained in Chapters 3-6. It is our intention to relate our results with other recent developments in feedback control design of uncertain dynamical systems. We present it chapter by chapter.

7.2.1 Summary and discussion of Chapter 3

The main result of this chapter was presented in Theorem 3.2. It was shown that for arbitrary admissible uncertainty realization $F \in \mathcal{F}$, the observer-feedback controlled system is ultimately bounded with respect to every Lyapunov ellipsoid containing the closed ball $\overline{I\!B}_n(\eta_1)$. In Lemma 3.1, we have proved the existence and continuation of solutions for the overall observer-feedback controlled system. Preceding that (in Theorem 3.1) we have established the existence of a stabilizing state feedback control by assuming the entire state is available for feedback purposes.

Our work here is an extension of that of Breinl and Leitmann (1983) in the directions which may be summarized as follows. First, we have used the Corless and Leitmann (1981) approach in the control design whereas they used Leitmann (1979b) approach. Secondly, we have generalized cone-bounded uncertainties to quadratically-bounded uncertainties. Thirdly, condition $TB = 0$ was imposed there whereas here we relaxed it to $\|TBg(\cdot)\| \leq$ constant. We remark from Kudva *et al.* (1980) that the condition $TB = 0$ holds if and only if rank CB = rank $B = m$ and transmission zeros of (C, A, B) is stable. Thus, the results obtained here are stronger than before. Furthermore, Lemma 3.1 provide the existence and continuation of solution of observer-feedback controlled system, which has not given earlier.

7.2.2 Summary and discussion of Chapter 4

We have proposed a new method of design of stabilizing dynamic output feedback of a class of uncertain systems. This was accomplished by initially considering "hypothetical" output y_h and then (in Theorem 4.1) a stabilizing static output feedback for hypothetical system was established by using the Steinberg and Ryan (1986) approach (fundamentally based on Barmish, Corless and Leitmann 1983). Then, the static output feedback was approximated by a realizable dynamic compenstor which filters the actual output y, and by using singular perturbation analysis akin to that Saberi and Khalil (1984) and Corless *et al.* (1989), it has been shown (in Theorem 4.2) that the feedback controlled system is globally uniformly asymptotically stable provided that the filter dynamics are sufficiently fast. A calculable threshold measure of fastness was provided (in Theorem 4.2).

By an analogous approach, we have generalized the proposed control design to include more general systems (i.e. to allow for additional uncertainties) by admitting a nonlinear discontinuous control component, modelled by an appropriately chosen set-valued map, and the overall controlled system consequently interpreted in the generalized sense of a controlled differential inclusion (Aubin and Cellina 1984). The additional structure on the uncertain function g were imposed in A4.4 and A4.5, and equivalent results were stated in Theorem 4.3 for static case, and in Theorem 4.4 for dynamic compensator case.

Our work here has been inspired by that of Steinberg and Ryan (1986) who suggested that their approach may be feasible for the case $r > 2$. It is our aim to extend their approach to multivariable version and to the cases $r \geq 2$. Case $r = 1$ turned out to be our special case.

In the discontinuous case, we generalized the Ryan (1988) approach to the case $r \geq 2$ with the help the results of Leitmann and Ryan (1987) on the decomposition of g.

7.2.3 Summary and discussion of Chapter 5

In this chapter, we have developed a stabilizing adaptive control, which mainly to circumvent the inherent conservatism induced by the crude estimates in a "worst case" design occured in Chapter 4. Moreover, it is applicable to the case for which bounds on the uncertainties may be unknown (i.e. to allow for bounded uncertainties with unknown bounds (Corless and Leitmann 1983, 1984)).

Our initial result contained in Theorem 5.1 where we have looked at a special case, i.e. $r = 1$. By Lyapunov analysis, it was shown that the adaptively controlled system exhibits the properties of universal adaptive stabilizer. For cases $r \geq 2$, we first proved Lemma 5.1 which is the non-autonomous version of Mårtensson's Lemma (Mårtensson 1986). Then, by using this lemma, we proved Theorem 5.2 which is our main result in adaptive control for the linear case. However, further conditions were imposed on g in order to apply the lemma.

Adaptive strategy is then generalized by expanding the class of allowable uncertainties. We developed an associate generalized adaptive output feedback strategy which is in the spirit of Ryan (1988) and akin to that of Mårtensson (1986), i.e. we expand to the cases $r \geq 2$ by using Mårtensson's method. However, this generalization is achieved at expanse of extra assumptions on the uncertain function g which is given in A5.2. In this discontinuous case, we first established Lemmas 5.3 and 5.4 (Lemma 5.4 is generalized non-autonomous version of Mårtensson's lemma). Then, by using these lemmas we proved the main result for the discontinuous case, which is given in Theorem 5.3.

Finally, we gave an example (a Maglev vehicle model) to illustrate the application of the proposed control design (linear case only).

7.2.4 Summary and discussion of Chapter 6

We addressed here the problem of designing static output feedback for a class of uncertain "relative degree 2" systems. In the first part, the approach undertaken is similar to that of Chapter 4, to show that there exist a stabilizing static output feedback control and was established in Theorem 6.1. Then, since the design is based on "worst case" analysis, we also conjectured an adaptive version of the static output feedback control by using a similar approach to Chapter 5 and was stated in Conjecture 6.1.

Our main aim here was to extend Morse (1985) and Steinberg and Ryan (1986) works to multivariable case and to avoid of using of dynamic compensator in Steinberg and Ryan (1986). This is done by imposing an extra set of assumptions which was given in A6.1. However, as might be expected, the structural properties on uncertainties are more restrictive as stated in A6.2 and A6.3.

7.3 Suggestions for future work

We briefly indicate here some possible extensions of our work which might be pursued, or some directions in which the work can be extended, in response to the recent trends in feedback design (see, for example, Kokotović 1985, DeCarlo *et al.* 1988 and Ljung 1988 for surveys) and in context of deterministic control of uncertain systems.

7.3.1 Observer-based design

One of possible direction in which our work might be extended is *non-linear observers*. Recently, this field of research has attracted many researchers, see for example, Walcott *et al.* (1987). This field may be subdivided into: *exact linearization* (Hunt *et al.* 1983, Su 1983) which transforms the original non-linear system into an equivalent linear system, *observers with linearizable error dynamics* (Krener and Respondek 1985, Respondek 1985) and *variable structure system observers* (Walcott and Żak 1987). Since our design has close links with variable structure system theory, the latter is a promising area of extension (see reference cited above and recent paper by DeCarlo *et al.* (1988)).

One of the problems that arose in this design is that γ_2 is required to be sufficiently small. One way to overcome this is to select it in optimal manner. An approach based on the stability radius of Hinrichsen and Pritchard (1986a, b) may be appropriate.

7.3.2 Dynamic compensator-based design

A recent development in singular perturbation theory is the use of *geometric methods* (see, e.g. Kokotović 1985). Our work might be extended in this framework, in particular along the lines of Khorasani and Kokotović 1987 and Shakey and O'Reilly 1987. Moreover, since our singular perturbation analysis is akin to Saberi and Khalil (1984), other possible direction is via composite control (see, e.g. Saberi and Khalil 1985); this approach has been used recently by Garofalo (1988).

7.3.3 Adaptive-based design

For this design, the possibility of using others adaptation laws is very promising, for example, adaptation laws of Ilchmann *et al.* (1987). Since universal adaptive stabilization is an active area of research recently, and the problem still far from complete (see Helmke and Prätzel-Wolters 1988), exploring further other adaptation laws along these lines is warranted.

7.3.4 Static output-based design

Certainly, some generalization could be done in this design, since only a few papers have appeared for "relative degree 2" systems (Morse 1985 and Steinberg and Ryan 1986), but the first task is to prove Conjecture 6.1 along the lines indicated.

One of possible extension to this design is to relax some assumptions. In particular, assumptions A6.1(ii)-(iii) could be replaced by condition rank $[CB \vdots CAB] = m$ and modifying A6.1(iv) accordingly. Tentative work in this direction suggests that, using a particular state transformation, stabilizing static output feedback is feasible. Then, an adaptive version might also be developed.

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